Birla Institute of Technology and Science, Pilani

1st Semester 2015-16

Algebra I (MATH F215)

Comprehensive Examination

Part A (Closed book)

Max. Time: 105 mins **Max. marks : 50**

Notation : (a) Use the following notation.

$\mathbb{Z} =$ the set of all integers, $\mathbb{Q} =$ the set of all rational numbers,

$\mathbb{R} =$ the set of all real numbers, $\mathbb{Z}_n =$ the set of all integers mod n

(b) In the product of permutations, left permutation operates first.

(c) For all other notations follow the text book.

1. (a) Express the permutation $(4,3,2)(5,3,2,1)(3,1)$ as product of disjoint cycles.

   (b) Write the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 5 & 6 & 3 & 2 \end{pmatrix}$ as product of transpositions and hence decide if it is even or odd permutation. [3+5]
2. Find the center of the symmetric group S_n of degree $n \geq 3$. Justify. [7]
3. State the class equation of a finite group G. Using it, prove that if G is a group of order $p^n > 1$ for a prime number p then the center of G contains a nonidentity element. [6]
4. Define the ring of real quaternions. Write $(1 - 2i + k)(4i + 5j + k)^{-1}$ as $a + bi + cj + dk$ where $a, b, c, d \in \mathbb{R}$. [4]
5. Show that a nonzero commutative ring R with a unit element is a field if and only if its only ideals are $\{0\}$ and R. [6]
6. Let R be a commutative ring and let

   $N = \{r \in R : r^n = 0$ for some positive integer $n\}$.

   Show that N is an ideal of R. [7]
7. Define a Euclidean ring R. Show that in a Euclidean ring R, every ideal is principal. [7]
8. In the Euclidean ring $\mathbb{Z}[i] = \{a + bi \in \mathbb{C} : a, b$ are integers$\}$, find the remainder when (a) $11 + 7i$ is divided by $18 - i$, (b) $18 - i$ is divided by $4i$. [5]

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Birla Institute of Technology and Science, Pilani

1st Semester 2015-16

Algebra I (MATH F215)

Comprehensive Examination

Part B (Open book)

Max. Time: 75 mins **Max. marks : 40**

Notation : (a) Use the following notation.

$\mathbb{Z} =$ the set of all integers, $\mathbb{Q} =$ the set of all rational numbers,

$\mathbb{R} =$ the set of all real numbers, $\mathbb{Z}_n =$ the set of all integers mod n

(b) In the product of permutations, left permutation operates first.

(c) For all other notations follow the text book.

1. Let G be a group with the identity element e and let $a, b, c \in G$. Let n be a positive integer such that $(abc)^n = e$. show that $(bca)^n = e$. [5]
2. Let $n > 2$ and $H = \{\sigma \in S_n : (1)\sigma = 1\}$. Show that H is a subgroup of S_n. Further let $H^* = \{ \sigma \in S_n : (n)\sigma = n\}$. Is there $\tau \in S_n$ such that $H^* = \tau H \tau^{-1}$ Justify. [7]
3. Let $\mathbb{Z}, \mathbb{Q}$ denote the additive groups of all integers and rational numbers respectively. Show that any element of $\mathbb{Q}/\mathbb{Z}$ has finite order. [5]
4. Let G be a finite group and $f: G \longrightarrow \mathbb{Z}$ be a homomorphism of groups. Prove that for all $x \in G$, $f(x) = 0$. [5]
5. Let $GL(2, \mathbb{R})$ denote the multiplicative group of all nonsingular 2x2 matrices with real entries. Find the centralizer of $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ in $GL(2, \mathbb{R})$. [6]
6. Recall that a subgroup H of a group G is called a characteristic subgroup of G if $\varphi(H) = H$ for any automorphism φ of G. Show that if H is a normal subgroup of a group G and K is a characteristic subgroup of H then K is a normal subgroup of G. [6]
7. Show that a simple group of order 60 can't have a subgroup of order 20. [6]

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