Birla Institute of Technology and Science, Pilani
First Semester 2015-2016
Algebra I (MATH F215)
Mid-Semester Exam (Closed Book)

Max. Time: 90 Minutes **Max. Marks: 70**

---

1. Let $(G, \cdot)$ be a group.

(i) If $(a \cdot b)^2 = a^2 \cdot b^2$ for all $a, b \in G$, then show that G is abelian. **[3]**

(ii) If, for all $a, b \in G$, $(a \cdot b)^i = a^i \cdot b^i$ for three consecutive $i \in \mathbb{Z}$, then show that G is abelian. **[7]**

2. Let G be an abelian group and let $a, b \in G$ be such that their orders $o(a)$, $o(b)$ are both finite and coprime to each other. Find $o(ab)$ in terms of $o(a)$ and $o(b)$. **[6]**

3. If H is any subgroup of a cyclic group G, then show that H is cyclic. **[8]**

4. If H, K are finite subgroups of a group G, then show that $o(HK) = \frac{o(H)o(K)}{o(H \cap K)}$. **[8]**

5. Define Euler's ϕ function and state Euler's theorem. Use it to show that, for an integer k, if $k^{27} \equiv 1 \bmod 100$, then $k \equiv 1 \bmod 100$. **[7]**

6. Define the kernel, $\ker(f)$, of a group homomorphism $f : G \to \bar{G}$. Show that $\ker(f)$ is a normal subgroup of G. **[6]**

7. Define the center $Z(G)$ and the inner automorphism group $I(G)$ of an arbitrary group G. Show that $I(G) \approx \frac{G}{Z(G)}$. **[10]**

8. Let M, N be both normal subgroups of a group G such that $M \cap N = \{e\}$. Show that $M \subseteq C(N)$, where $C(N)$ denotes the centralizer of N in G. **[7]**

9. Let H be a subgroup of a group G, S be the set of all right cosets of H in G and $A(S)$ be the group of all permutations of S. Construct a group homomorphism $\theta : G \to A(S)$ such that $\ker \theta = \bigcap_{g \in G} g^{-1}Hg$. Justify your answer. **[8]**