BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE, PILANI
First Semester 2015 – 16
MATH F312 : Ordinary Differential Equations
Comprehensive Examination

Duration : 3 hrs   Date : 01/12/2015   MM : 90

Part-A: Closed book   **Max Marks: 20**   **Max time: 40 mins**   SET A

Name: **ID No.:**

Note: Write only the correct options (A, B, C or D in CAPITAL LETTER ONLY) of the given questions in the box provided below. Each correct answer carries 2 marks and incorrect answer carries -0.5 mark. Over writing/cutting will be treated as unattempted.

Q. No.	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Answer	D	D	A	B	C	A	A	B	B	C

A1. Let $u(t)$ be a nonnegative functions satisfying $u(t) \le \int_0^t Ku(s)ds;\ K > 0, 0 \le t \le 1.$ Then $u(t)$ is given by

(A) $u(t) = e^{Kt}$ (B) $u(t) = Ke^t$
(C) $u(t) = -Ke^t$ (D) none of these

A2. Let $u_1(t)$ *and* $u_2(t)$ be any two solution of the differential equation: $u'' - u' + Q(t)u = 0$, $x \epsilon I = [0,\ \infty)$, where $Q(t)$ is continuous on I. Let $W(t)$ be the Wronskian of $u_1(t)$ *and* $u_2(t)$ at any point $t\epsilon I$ with $W(0) = 1$. Then which of the following statements is TRUE?

(A) $W(t)$ is decreasing function of t and $\max_{t\in I} W(t) = 1.$
(B) $u_1(t)$ *and* $u_2(t)$ are linearly dependent solutions of the given differential equation.
(C) $W(t) = e^{-t}$.
(D) $W(t) = e^t$.

A3. Let $u(t)$ be any nontrivial solution of $u''' + 5u'' + 2u' + 7u = 0$. Then which of the following statements is true?

(A) $\lim_{t\to\infty} |u(t)| = 0$ (B) $\lim_{t\to\infty} |u(t)| = \infty$

(C) $\lim_{t\to\infty} |u(t)| = 1$ (D) none of these

A4. The zero solution of the differential equation $u' = u(u^2 - 1)$ is

(A) stable but not asymptotically stable   (B) asymptotically stable
(C) unstable   (D) none of these

A5. The equilibrium point $(-1,-1)$ of the system of differential equations of

$$x_1' = x_1 - x_2,\ x_2' = x_1^2 + x_2^2 - 2 \text{ is}$$

(A) stable but not asymptotically stable
(B) asymptotically stable
(C) a saddle point
(D) none of these

A6. Let $\phi(t)$ be continuous on $[0,\infty)$, and $\phi(t) \to 0\ as\ t \to \infty$. Let $u(t)$ be a non trivial solution of the differential equation $u'' + (1+\phi(t))\mathrm{u} = 0$. Then which of the following statements is true?

(A) $u(t)$ is oscillatory
(B) $u(t)$ is non-oscillatory
(C) $u(t)$ is oscillatory in $0 \le t < 1$, and non-oscillatory outside the interval $[0,1)$
(D) none of these

A7. Let $u(t)$ be any arbitrary solution of the differential equation $u'' + 5e^t\, u = 0$. Then which of the following statements is true?

(A) $u(t)$ is bounded
(B) $u(t)$ is unbounded
(C) $u(t)$ is bounded in $(-1,1)$, but unbounded outside $(-1,1)$
(D) none of these

A8. Let all the solutions of the differential equation

$$u'' + \left(1 + \frac{\cos^2 t}{t^a}\right) u = 0, \qquad t > 0$$

be bounded. Then the value(s) of a is(are)

(A) $a = 1$
(B) $a > 1$
(C) $a < 1$
(D) none of these

A9. The limit cycle $C: x_1^2 + x_2^2 = 1$ of the model system $x_1' = -x_2,\ x_2' = x_1 + x_2(x_1^2 + x_2^2 - 1)$ is

(A) stable
(B) unstable
(C) semi-stable
(D) none of these

A10. For the system $x' = A(t)x$, where

$$A(t) = \begin{pmatrix} -1 & e^{2t} \\ 0 & -1 \end{pmatrix},$$

the solution $x(t) = 0$ is

(A) a stable focus
(B) a spiral node
(C) unstable
(D) none of these

*****END OF PART A*****

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Name: **ID No.:**

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Q. No.	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Answer	B	B	C	D	D	A	B	C	A	A

A1. Let all the solutions of the differential equation

$$u'' + \left(1 + \frac{\cos^2 t}{t^a}\right)u = 0, \qquad t > 0$$

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A2. The limit cycle $C: x_1^2 + x_2^2 = 1$ of the model system $x_1' = -x_2,\ x_2' = x_1 + x_2(x_1^2 + x_2^2 - 1)$ is

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A3. For the system $x' = A(t)x$, where

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A5. Let $u_1(t)$ *and* $u_2(t)$ be any two solution of the differential equation: $u" - u' + Q(t)u = 0$, $x \epsilon I = [0,\ \infty)$, where $Q(t)$ is continuous on I. Let $W(t)$ be the Wronskian of $u_1(t)$ *and* $u_2(t)$ at any point $t\epsilon I$ with $W(0) = 1$. Then which of the following statements is TRUE?

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(D) $W(t) = e^{t}$.

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(A) $\lim_{t\to\infty} |u(t)| = 0$

(B) $\lim_{t\to\infty} |u(t)| = \infty$

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A9. Let $\phi(t)$ be continuous on $[0, \infty)$, and $\phi(t) \to 0\ \ as\ t \to \infty$. Let $u(t)$ be a non trivial solution of the differential equation $u" + (1 + \phi(t))\mathrm{u} = 0$. Then which of the following statements is true?

(A) $u(t)$ is oscillatory

(B) $u(t)$ is non-oscillatory

(C) $u(t)$ is oscillatory in $0 \le t < 1$, and non-oscillatory outside the interval $[0, 1)$

(D) none of these

A10. Let $u(t)$ be any arbitrary solution of the differential equation $u'' + 5e^t\ u = 0$. Then which of the following statements is true?

(A) $u(t)$ is bounded

(B) $u(t)$ is unbounded

(C) $u(t)$ is bounded in $(-1, 1)$,but unbounded outside $(-1, 1)$

(D) none of these

END OF PART B

MATH F312 : Ordinary Differential Equations
Comprehensive Examination, 1st Sem 2015-16

PART-B: Closed Book	Max Marks: 30	Max time: 60 mins

B1. Show that all the solutions of $u^{''} + e^t u = 0$ are bounded on $[0, \infty)$ as $t \to \infty$. Is the differential equation $u^{''} + e^t u = 0$ oscillatory on $[0, \infty)$? Justify your answer. [4+4]

B2. Let $A(t)$ be an $n \times n$ constinuous matrix on $[0, \infty)$, and x is an n-vector. Show that if all the solutions of $x^{'} = A(t)x$ are stable, then they are also bounded. [8]

B3. Consider the IVP:

$$u^{'} = tu - e^{-t}u^2, \;\; u(0) = 1 = u_0,$$

on $\Omega: 0 \le t \le 2, \;\; |u - u_0| \le 1.$

Then estimate the variation in its solution for $0 \le t \le 1$ if u_0 is perturbed by 0.01. [8]

B4. Show that all the solutions of the differential equations

$$u^{''} + \left(1 + \frac{2}{t(t^2+1)}\right)u = 0, \qquad t > 0$$

are bounded on $[1, \; \infty)$. [6]

***** End of Part B*****

MATH F312 : Ordinary Differential Equations
Comprehensive Examination, 1st Sem 2015-16

PART-C:Open Book

Max Marks: 40 | Max time: 80 mins

C1. Convert the scalar differential equation $u''' + au'' + bu' + cu = w(t)$, in the form of a vector differential equation. Assuming that a, b, c are constants satisfying $a > 0, c > 0$ and $ab > c$, find the conditions on $w(t)$ such that all solutions of the above given differential equation are bounded on $[0, \infty)$. [10]

C2. Let $x(t)$ be any non-trivial solution of the vector-differential equation
$x' = Ax$, where

$$A = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -2 \end{pmatrix}, \; x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Then prove or disprove: $\lim_{t\to\infty} \|x(t)\| = 0$. [10]

C3. Consider the following model system

$$\frac{dx}{dt} = x(22 - 5\,lnx - 4\,lny),$$

$$\frac{dy}{dt} = y(43 - 8\,lnx - 9\,lny),$$

where x and y are greater than one. For the above model system, answer the following questions:

(i) Find a positive equilibrium $E^*(x^*, y^*)$, $(x^* > 0,\ y^* > 0)$. [2]

(ii) Linearize the above model around the equilibrium point $E^*(x^*, y^*)$. [4]

(iii) Compute the variational matrix at the equilibrium point $E^*(x^*, y^*)$, and then test the stability of the positive equilibrium point $E^*(x^*, y^*)$ by eigenvalue method. [4]

C4. For the following food-web model system (the species of density $x(t)$ is a food for another species of density $y(t)$, and the species of density $y(t)$ is a food for another species of density $z(t)$)

$$\frac{dx}{dt} = 6x - x^2 - xy,$$
$$\frac{dy}{dt} = -5y + 4xy - yz,$$
$$\frac{dz}{dt} = -6z + 2yz,$$

answer the following questions:

(i) Find the positive equilibrium point $E^*(x^*, y^*, z^*)$ for the given system of equations. [3]

(ii) Taking the positive definite function
$V(x(t), y(t), z(t)) = \left(x - x^* - x^* \ln\frac{x}{x^*}\right) + a\left(y - y^* - y^* \ln\frac{y}{y^*}\right) + \ b\left(z - z^* - z^* \ln\frac{z}{z^*}\right),$
and by choosing the appropriate positive values of the constants a and b, prove that the equilibrium point $E^*(x^*, y^*, z^*)$ is non-linearly asymptotically stable in the interior of the positive orthant of the xyz −space. [7]

***** END OF THE QUESTION PAPER*****