BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE, PILANI
SECOND SEMESTER 2021-22
BITS F316: *Nonlinear Dynamics and Chaos*
Comprehensive Examination (Part B)
Open Book

Total marks: 60 Time: 120 mins

Hamiltonian systems

1. Is the following transformations between (q, p) to (Q, P) canonical?

$$Q = \ln \frac{\sin p}{q}, \qquad P = q \cot p \quad [2]$$

2. Consider a particle of mass m bouncing vertically and elastically on a flat surface. Assuming acceleration due to gravity g to be constant and a total energy to be E

   (a) Write down the Hamiltonian in terms of the usual variables (z, p) where z is the vertical displacement and p is the momentum [2]

   (b) Draw the phase space in the variables (z, p) and also in action and angle variables (θ, I) [2]

   (c) What is the time period of motion T ? How does T scale with the total energy? [3]

   (d) Write down the angle variable in terms of the old canonical variables. [3]

Maps: Fixed points, Chaos

3. Consider the Logistic map on the unit interval.

$$x_{n+1} = r x_n (1 - x_n)$$

Calculate explicitly the fixed points of the map and the period two cycle for $r = 3.5$. [4]

4. Consider the quadratic map

$$x_{n+1} = x_n^2 + c$$

   (a) Find the fixed points and discuss stability and bifurcation with the parameter c.

   (b) Show that the logistic map $x_{n+1} = r x_n (1 - x_n)$ can be transformed to the quadratic map above by a linear transformation $x_n = a y_n + b$. Find a and b.

   (c) Find the probability density $\rho(x)$ for the quadratic map from the known probability density of the logistic map for $r = 4$. [9]

5. If a map has a Lyapunov exponent $\lambda = \ln 5$, after what time an error in initial condition $\epsilon = 10^{-8}$ amplify to a maximum tolerance of 10^{-2}.

Consider a map on the unit cube

$$\Gamma : (x_n, y_n, z_n) \to (x_{n+1}, y_{n+1}, z_{n+1}) = (2x_n + y_n, x_n + y_n, 2z_n)\text{mod}1$$

Is this map volume preserving ? Will this map exhibit recurrence ? Is the map chaotic ? (give brief reasons qualitative/quantitative) [2 + 3]

Probability measure and entropy

6. For some one dimensional map $f : [0, 1] \to [0, 1]$, the equilibrium invariant probability density $\rho(x)$ given by

$$\rho(x) = 2(1 - x) \quad x \in [0, 1]$$

What is the entropy S if the interval is divided into three equal partitions. [4]

7. Consider the following one dimensional map

$$x_{n+1} = \frac{5}{3}x_n \text{ for } 0 \le x_n \le \frac{3}{5} \quad \& \quad x_{n+1} = 2x_n - \frac{6}{5} \quad \text{for} \quad \frac{3}{5} < x_n \le 1$$

   (a) Write down the Frobenius-Perron invariant probability density $\rho(x)$as a functional relationship and find the normalized $\rho(x)$ in the entire interval by assuming that $\rho(x)$ constant in each of the linear pieces of the map. Plot the invariant probability density. [4]

   (b) By calculating the Lyapunov exponent of the map make a statement about its long term dynamics? [2]

Symbolic dynamics

8. (a) Consider a map of the unit interval

$$x_{n+1} = 4x \text{ mod } 1$$

   If we partition the unit interval into 4 equal partitions $[0, 1/4)$, $[1/4, 1/2)$, $[1/2, 3/4)$ and $[3/4, 1]$ which we call A, T, C and G respectively. what initial condition x_0 will generate the following time itinerary of symbols $CATGAAATCACG$ [4]

   (b) Under the usual tent map of the unit interval (partitioned into L and R) what is the time itinerary of L and R if we start from the initial condition $x_0 = 0.110101100101$? [2]

9. The Gauss map $G : [0, 1] \to [0, 1]$ is the following map

$$G(x) = 0 \qquad if\ x = 0$$
$$= \left(\frac{1}{x}\right) \text{mod } 1 \quad if\ 0 < x \le 1$$

The map is discontinuous for $x = 1/n$, where n is an integer $n = 1, 2, 3....$

Every real number x in the interval can be expressed as a continued fraction

$$x = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + ...}}}$$

We shall write $x \in [0, 1]$ as $(a_0, a_1, a_2....)$. Thus we say that $(a_0, a_1, a_2....)$ is the continued fraction representation for x. For rational numbers, the sequence would terminate while it would not terminate for irrational numbers.

   (a) Show that $G((a_0, a_1, a_2....))$ is a shift map in the continued fraction representation of the numbers in $[0, 1]$

   (b) Find the largest fixed point of the map

   (c) Find the Lyapunov exponent $\lambda(x_0)$ where $x_0 = (2, 5, 1, 2, 1, 1, 1, 1, 1, 1, ...)$

   (d) Write any intial condition x in a continued fraction representation which falls is on a period 2 orbit. [1+ 1 +1 +1 = 4]

Fractals

10. Consider a straight line segment of length 1 unit. In figure A the straight line is bent at two places (wedges are at right angles) to form three equal segments. If this process is continued on each of the newly formed straight line segments ad-infinitum one would have a fractal. What is the fractal dimension for this object? [4]

    In figure B, another construction starting from the unit line segment is shown. Each of the smaller line segments are equal in length. What is the fractal dimension for this? If the operation in figure B is applied repeatedly to the sides of a square island one would eventually have a fractal shoreline as shown. [3]

11. *In 1981 J. Avron and B. Simon demonstrated that the beautiful ring structure of Saturn is actually a fractal caused by gravitational resonances The rings along the radial direction had spacings which resembled the Cantor set. However, what they found was not a familiar Cantor set, such as the middle-thirds set, but rather a fat Cantor set, which unlike the Cantor set has some non-zero length.*

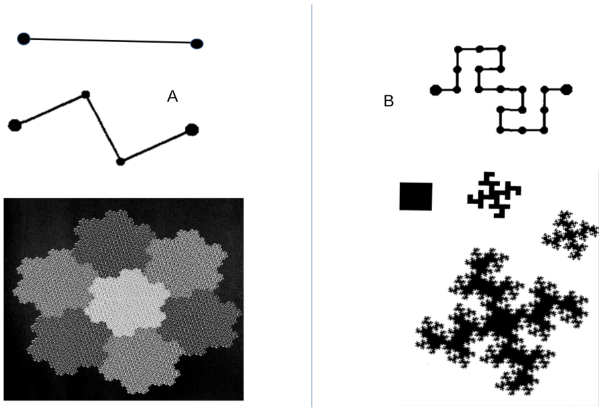

Figure 1: From "*The Fractal Geometry of Nature*" by Benoit B Mandelbrot, W. H. Freeman and Co.(1982)

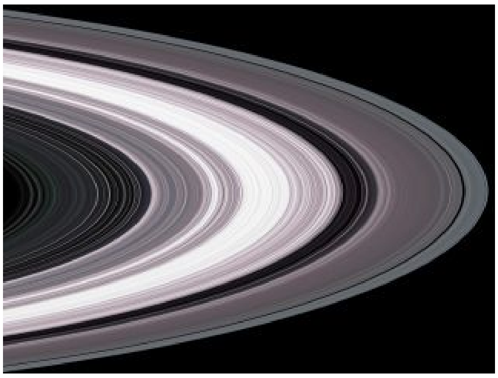

Figure 2: A simulated image based on some of the first data Cassini gathered after its arrival at Saturn in 2004. (Image: © NASA/JPL)

Construction:

Consider an interval $[0, 1]$.

At step 1, we remove an interval of length $1/3$ from the middle of the interval.

At step 2, we remove an interval of length $1/9$ from the middle of each of the two intervals remaining.

At step 3, we remove an interval of length $1/27$ from the middle of each of the four intervals remaining.

If we carry this on ad-infinitum, show that the set indeed is left with some finite length. [3]