BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE, PILANI
SECOND SEMESTER 2021-22
BITS F316: *Nonlinear Dynamics and Chaos*
Midterm test (Part B)
Open Book

Total marks: 40 Time: 60 mins

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1. *Oscillation in equilibrium chemistry !*
   The following chemical sequence from a substrate to a product $S \to P$ is given by

$$S + X \to 2X$$
$$X + Y \to 2Y$$
$$Y \to P$$

   If k_1, k_2 and k_3 denote the rate constants for the three reactions respectively, the rate equations for the concentrations $x = [X]$, $y = [Y]$ of the two intermediates X and Y are given by

$$\frac{dx}{dt} = (k_1[S] - k_2 y)x$$
$$\frac{dy}{dt} = (k_2 x - k_3)y$$

   (a) What are the equilibrium concentrations (x_0, y_0) of $[X]$ and $[Y]$ for a fixed substrate concentration $[S]$?
   (b) Discuss the stability of the fixed point and show that the concentration of the intermediates oscillate ! [2 + 6]

2. Consider a 3 dimensional dynamical system

$$\dot{x} = x + y^4, \quad \dot{y} = -y, \quad \dot{z} = -2z + y^3$$

   (a) Is the origin $(0, 0, 0)$ a Hyperbolic fixed point?
   (b) What is the 3×3 linearized stability matrix A?
   (c) Find the spaces E_U and E_S.
   (d) What is the behaviour of the flow in the $x - y$ plane around the origin?
   (e) What is the stable manifold W_S for this system? Is the relation between E_S and W_S for this system consistent with the stable-manifold theorem ? [1 + 2 + 2 + 1 + 2]

3. Consider the celebrated Lorenz system

$$\dot{x} = \sigma(y - x), \quad \dot{y} = rx - y - xz, \quad \dot{z} = xy - bz$$

   where r, σ, b are positive constants. What can be said about the stability of the origin from linear theory and from Lyapunov theory assuming a Lyapunov function $L = \alpha x^2 + \beta y^2 + \gamma z^2$ (assume $0 < r < 1$). [6]

4. Consider a dynamical system which may be written using the complex variable $z = x + iy$ as

$$\dot{z} = (z - 3i)(z + 3i)$$

   What are the fixed points and their stabilities? Draw a generic phase flow in the (x, y) plane and find the winding number for a closed contour which includes all the fixed points in its interior. [6]

5. For the following systems identify the fixed points/ limit cycles (if any) and discuss their stability

   (a)

$$\dot{x} = -y + f(r)x, \qquad \dot{y} = x + f(r)y \qquad \text{where} \quad r = \sqrt{x^2 + y^2}$$

   (b)

$$\dot{r} = r^3 - 3r^2 + 2r, \quad \dot{\theta} = 1$$

   (c)

$$\dot{\mathbf{r}} = \mathbf{r}(\alpha \sin \beta r^2)/r^2 + \gamma \left(x\hat{j} - y\hat{i} \right)$$

   where α, β, γ are positive constants and $\mathbf{r} = x\hat{i} - y\hat{j}$ and $r = |\mathbf{r}|$

   [4 + 4 + 4]