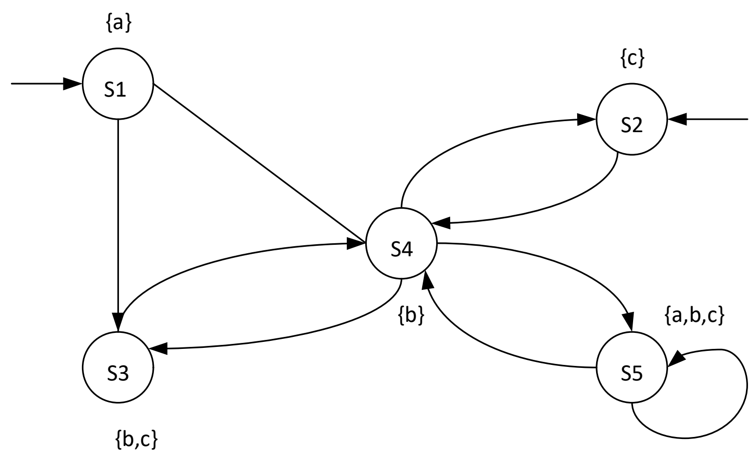
1. We can represent binary numbers as sequences of propositions, where a proposition represents 1 if it is assigned T and 0 otherwise. For instance, p1p2 represents 10 in the interpretation {p1 → T, p2 → F}. Give a propositional formula expressing that a binary number r1r2r3 is the sum of p1p2 and q1q2 (where p1, p2, q1, q2, r1, r2, r3 are propositions). **[8M]**

Solution: We assume that the least significant bit is indexed by the atom with the highest subscript (e.g. p2 is the least significant bit of p1p2). The addition of p1p2 and q1q2 can be formulated as the conjunction of following propositional statements. r3 ↔ (p3 = q3) , r2 ↔ ((p3 ∧ q3) ∧ (p2 ↔ q2)) ∨ (¬(p3 ∧ q3) ∧ (p2 = q2)) , r1 ↔ ((p3 ∧ q3) ∧ (q2 ∨ p2))

2. Consider the following transition system with the set of atomic propositions {a,b,c}: **[12 M]**



Indicate for each of the for each of the following LTL formula Φi whether TS |= Φi

a. Φ1 = EGc
b. Φ2= GEc
c. Φ3= X¬c → XXc
d. Ga
e. a U G(b∨c)
f. (XXb) U (b ∨ c)

3. Let S be a (finite or infinite) set of propositional formulas. Propositional Logic S is said to be satisfiable if there is a valuation v under which every formula in S holds. S is said to be finitely satisfiable if every finite subset T of S is satisfiable. Note that if S is satisfiable, then it is, of course, finitely satisfiable.

   A. Is the converse true? Prove your answer. **[10M]**
   B. Prove the following lemma: Let S be finitely satisfiable. Let α be any formula. Then either S ∪ {α} is finitely satisfiable or S ∪ {¬α} is finitely satisfiable.**[10M]**

4. Find all models of the formula $\psi = \neg(((p \rightarrow q) \leftrightarrow \neg p) \rightarrow p)$ **[10M]**
5. Prove that $\wedge$ and $\vee$ cannot define negation. **[10M]**

   The proof is by contradiction. Suppose that $\wedge$ and $\vee$ can define negation, i.e., supose $\neg\phi \equiv \phi X \dots X \phi$, for a finite number of applications of X, where X is either $\wedge$ or $\vee$. Let v be an interpretation that assign T to ϕ, then $F = v(\neg\phi) = v(\phi X \dots X\phi)$. Since X is either $\wedge$ or $\vee$, for $v(\phi X \dots X\phi)$ to be equal to F at least one of the operands must valuate to F. Since all operands are equal, and $\wedge$ and $\vee$ are associative, this requires that $v(\phi) = F$. But this contradicts the assumption that $v(\phi) = T$. Therefore our initial supposition that $\wedge$ and $\vee$ can define negation is false.

6. Prove the following theorem: $\{\psi 1, \dots, \psi n\} \models \phi$ if and only if $\models (\psi 1 \wedge \cdots \wedge \psi n) \rightarrow \phi$ **[10M]**