Birla Institute of Technology and Science, Pilani – K K Birla Goa Campus

2022-2023 Sem I	Comprehensive Exam	27-12-2022
CS F222	Discr. Struc. for Comp. Sc.	10:00-13:00 hrs
Marks 80	Weightage 40%	Mixed Mode, 180 min

Instructions:	*Assumptions must be clear and justified. Consistency is a must, conformance not enough.* ***No rough work inside answers. Write final answers in rectangles*** *and proofs below that in* ***numbered steps.***
$0 \in \mathbb{N}$	Rationals are $\mathbb{Q}$ and reals are $\mathbb{R}$.

Q1 (a) Given $x = +\sqrt{2}$ (the positive square root of 2), consider the set

$$S_x = \{\min(jx - \lfloor jx \rfloor, \lceil jx \rceil - jx) : \forall j \in \mathbb{N}\}.$$

Is it countable? Justify your answer in **no more than one line**. If more lines are seen it will yield negative marks. **[1+2=3]**

Q1 (b) Given x as above, let $S_{x,n} = \{\min(jx - \lfloor jx \rfloor, \lceil jx \rceil - jx) : \forall j \in \mathbb{N}, j \leq n\}$ and $a_n = |S_{x,n}|$. Find the counting sequence a_n, and prove your solution. **[5+5=10]**

Q1 (c) Solve (a) and (b) with $x = \frac{1}{7}$ instead of $\sqrt{2}$. **[1+1+2+2=6]**

Q2 (a) In how many ways can we select k objects from abundant supply of objects of n types, with unrestricted (but upto k) repetitions of any type? Justify your answer. **[10]**

Q2 (b) From the (a) part, find a recurrence between the successive numbers in the counting sequence $a_k = |\texttt{select}(\texttt{types} = 5, \texttt{objects} = \texttt{k})|$ for $n = 5$ and $k = 1, 2, 3, \ldots$. **[5]**

Q2 (c) Solve the recurrence and recover the original expression you had obtained in (a). Write the solution in numbered steps with each step in the left column and its justification in the right column. Clumsy and chaotic answers lead to forfeiture of rechecks. **[10]**

Q3. First 10 sequence numbers are given below of two sequences $\langle b_n \rangle, \langle c_n \rangle$ obtained in some counting process. Find the recurrences and solve them to give a closed form expression each for the n^{th} sequence member of each sequence. **[10+6=16]**

n	1	2	3	4	5	6	7	8	9	10
b_n	4	10	20	35	56	84	120	165	220	286
c_n	1	3	4	7	11	18	29	47	76	123

Step marks for this question depend on neatness and consistency in your answers.

Q4 (a) In the classical model of rabbits multiplying on an island, each pair produces a new pair every month after two months of age. It all begins with one just born pair put on the island. The only twist in the tale is that after 3 months of age (i.e. after giving birth for the second time) any pair can die with a probability $1 - \frac{2}{m}$ where m is their age in months. Find the minimum and maximum numbers of pairs on the island after 10 months, and their probabilities. **[10]**

Q4 (b) You can get one of the 4 rewards, of values ₹1, 2, 4, 8 hundred, by choosing to scratch one of the four scratch cards arrived on your smartphone on a pay app. But this is a two-step process. You first input a number (your choice of a card). If that number is of the ₹100 card, you are done, you get ₹100. Otherwise, you are shown the ₹100 card, that you had not chosen, and then you can switch or stick to your choice. Then you get the reward under your final choice. What will be your strategy (always switch, always stick, or switch with a probability)? Give a clear strategy and argue that it maximises the chance of getting the highest amount on average. **[10]**

Solution

Q1 (a) It is patently countable because it is a sequence – a mapping from $\mathbb{N}$.**[1+2=3]** **Q1 (b)** Since x is irrational, $j \neq k \Leftrightarrow \min(jx - \lfloor jx \rfloor, \lceil jx \rceil - jx) \neq \min(kx - \lfloor kx \rfloor, \lceil kx \rceil - kx)$ for $j, k \in \mathbb{N}$ because otherwise x would be a ratio of difference between two floor/ceilings and $j - k$ (rational). Thus, for each n, $S_{x,n}$ will have $n+1$ unique irrationals, for $j = 0, 1, \ldots, n$. Thus, $a_n = n + 1$. **[5:proof of uniqueness, 4:n or $n+1$. 1:detecting $n+1$, not n]**

Q1 (c) With $x = \frac{1}{7}$ (rational), it is a finite periodic sequence given below.

j	$\min(jx - \lfloor jx \rfloor, \lceil jx \rceil - jx)$
0	0
1	0.142857142857...
2	0.285714285714...
3	0.428571428571...
4	0.428571428571...
5	0.285714285714...
6	0.142857142857...
7	0

Basically, $j \equiv k \mod 7$ or $j \equiv |k - 7| \mod 7$ will make the corresponding sequence numbers equal. Thus there are exactly 4 members in the sequence, and $a_n = \min(n+1, 4)$. Finite means countable, no need to state explicitly.
In marks distribution, exact marks are given (the table is not needed): **[1+1+2+2=6]**

Q2 (a) Selecting k objects from abundant supply of n types is like inserting partitions between groups, groups not more than n, among k people:

$$\checkmark\ \checkmark\ \checkmark \mid \checkmark\ \checkmark\ \checkmark\ \checkmark \mid \checkmark\ \checkmark \mid \checkmark \mid \checkmark$$

One selection of 11 objects of 4 or more types.

$$\mid \checkmark\ \checkmark\ \checkmark \mid \checkmark\ \checkmark\ \checkmark\ \checkmark \mid \checkmark\ \checkmark \mid \checkmark \mid \checkmark \mid \ \mid$$

Like, the same selection with 7 types. 3 types are absent in this selection.

Thus, it is like selecting $n - 1$ positions among $n + k - 1$ (the -1 is there because at least one type as to be assigned). So the count is

$$\binom{n+k-1}{n-1} = \binom{n+k-1}{k}.$$

Justifications for this may be written in many different ways. It is like allotting k votes among n candidates, with possible zeros to some candidates. (If $n < k$ then some candidates must get 0 votes.) It is the number of monotonic (nondecreasing or nonincreasing) sequences of length k among positive numbers $a_1 \leq a_2 \leq \cdots \leq a_n$. This one is more likely to be written by our students. Another explanation is distributing n cookies to k kids, again some may get 0 cookies. It is the number of unordered partitions of k in $\leq n$ parts.
Some students may prove by induction: double induction on n, k:

Base Both $n = k = 1$, obvious. Again $n > 1, k = 1$ then $\binom{n+1-1}{n-1} = \binom{n+1-1}{1} = n$; $n = 1, k > 1$ then $\binom{1+k-1}{1-1} = \binom{1+k-1}{k} = 1$.

Step $n \to n+1$, k arbitrary By the I.H. for n types the count is $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$. Now when we add a type, we have to add to the count all the ways of selecting j, $0 < j \leq k$ objects of the new type, with the remaining $k-j$ objects of n types. Recall we assumed the I.H. for arbitrary (or all) k. Thus the new count is

$$\binom{n+k-1}{k} + \sum_{j=1}^{k} \binom{n+k-j-1}{k-j} = \sum_{j=0}^{k} \binom{n+k-j-1}{k-j} = \sum_{j=1}^{k} \binom{n+j-1}{j}$$

and, using the known binomial identities, we get the RHS to be exactly $\binom{n+k}{k+1}$.

Step $k \to k+1$, n arbitrary For the same number of types, if we add one more object, then each earlier selection can be extended with one more object each of one of the types from the first type to the type of the first object in the selection ordered in non-decreasing order of type indices. Thus the new count is $\sum_{m=1}^{n} \binom{m+k-1}{m-1}$ again yielding $\binom{n+k}{k+1}$.

Q2 (b) The recurrence between the successive numbers in the counting sequence $a_k = |\texttt{select}(\texttt{types} = 5, \texttt{objects} = \texttt{k})|$ for $n = 5$ is obtained thus:
$a_{k+1} = \binom{k+5}{k+1} = \frac{(k+5)!}{(k+1)!4!} = \frac{k+5}{k+1} \times \frac{(k+4)!}{k!4!} = \frac{k+5}{k+1} a_k$.
Q2 (c) Solving $a_{k+1} = \frac{k+5}{k+1} a_k, a_0 = 1, a_1 = 5$ by recursion thus:

$$a_{k+1} = \frac{k+5}{k+1} a_k = \frac{k+5}{k+1} \times \frac{k+4}{k} a_{k-1} = \overset{k \text{ times}}{\cdots} = \prod_{j=1}^{k+1} \frac{j+4}{j} = \frac{\prod_{j=5}^{k+5} j}{(k+1)!} = \frac{(k+5)!}{(k+1)!4!} = \binom{k+5}{k+1}$$

There can be other ways, but I have shown the simplest and self-explanatory one.

Q3.

$$b_n = \langle \quad 4 \quad 10 \quad 20 \quad 35 \quad 56 \quad 84 \quad 120 \quad 165 \quad 220 \quad 286 \quad \rangle$$

One way to find a recurrence is trying ratios. Thus,

$$\tfrac{b_{n+1}}{b_n} = \langle \quad 5/2 \quad 2 \quad 7/4 \quad 8/5 \quad 3/2 \quad 10/7 \quad 11/8 \quad 4/3 \quad 13/10 \quad \rangle$$

The minimal representation of the ratios is misleading, still the pattern is unmistakable: $\frac{b_{n+1}}{b_n} = \frac{n+4}{n}$. So if you detect the similarity with 2(c) above, it is "`select(types=4,objects=n)`". Of course, without detecting that also, by the similar recursion, we get the formula. Another method is trying differences: $\Delta b = 6\ 10\ 15\ 21\ 28\ 36\ 45\ 55\ 66$; $\Delta^2 b = 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11$; then it is constant 1. Thus, the third derivative is constant and the fourth zero: the sequence must be cubic. Guess

$$\begin{aligned} b_n =& An^3 + Bn^2 + Cn + D \\ 4 =& A + B + C + D \\ 10 =& 8A + 4B + 2C + D \\ 20 =& 27A + 9B + 3C + D \\ 35 =& 64A + 16B + 4C + D \end{aligned}$$

and, solving these: $b_n = \frac{n^3}{6} + n^2 + \frac{11n}{6} + 1$. If one further factorises this and recovers $b_n = \binom{n+3}{3}$ well, one is a genius. But without that also, we should give credit. In fact, if anyone does follow this approach, then one would most likely stop at the numericals: $b_n = 0.167n^3 + n^2 + 1.83n + 1$ ... but this also should get full credit.
In the second sequence,

$$c_n = \langle \quad 1 \quad 3 \quad 4 \quad 7 \quad 11 \quad 18 \quad 29 \quad 47 \quad 76 \quad 123 \quad \rangle$$

the recurrence is obvious: $c_{n+2} = c_{n+1} + c_n$ and $c_1 = 1, c_2 = 3$. Solving it like we do the Fibonacci sequence and suchlike, we get $\alpha_{1,2} = \frac{1 \pm \sqrt{5}}{2}$ and $A = B = 1$ and $c_n = \varphi^n + (1 - \varphi)^n$ where $\varphi = \frac{1+\sqrt{5}}{2}$, the Golden Ratio. This being simpler, there are only 6 marks.

Q4 (a) Minimum pairs remain when every pair dies at the end of 3 months of age, producing 2 new pairs before dying. Modelling this figuratively, we get 16 pairs remaining and 11 dying (three of them dying at the end of 10 months) Some fastidious textbook fans want to omit just-borns from counts. But again, those insisting on not counting just-borns should also not discount the just-dead, which are 3 in our case. So then the minimum for those fastidious people not counting things happening at 00:01 hrs on the 11th day should answer minimum as 12. **What is unchanged is the minimum's probability**
$\left(\frac{1}{3}\right)^{11} = 5.64 \times 10^{-6}$.
Maximum pairs are F_{11} (because we begin $F_0 = 1, F_1 = 1, F_2 = 2$ instead of $F_0 = 0$) with probability that no pair dies. That probability is the probability that no pair dies ever. For that probability, note that F_{n-1} pairs are born at the end of n months, each defying death with probabilities $\frac{2}{m}$ for $m = 3, 4, ..., 10 - n$. Also include the exceptional first pair, defying death from months 3 to 10. Thus the total becomes

$$\prod_{n=3}^{10} \left(\frac{2}{n}\right) \prod_{n=2}^{10} \left(\prod_{m=3}^{10-n} \left(\frac{2}{m}\right)\right)^{F_{n-1}} = \frac{512}{3628800} \times 9.55 \times 10^{-12} = 1.347 \times 10^{-15}$$

which is infinitesimally small, so no point in expecting to hit it in simulation probabilistically.

	With Just-borns	No Just-borns	Add Now-dying	Probab
Min	16	9	12	5.64×10^{-6}
Max	89	55	55	1.347×10^{-15}

Q4 (b) The probabilities of winning each reward, given the strategy, are enumerated in the table below.

First Choice	Second Choice	Reward	Probability
100	100	100	0.25
200	200	200	0.25
400	400	400	0.25
800	800	800	0.25
200	400	400	0.125
200	800	800	0.125
400	200	200	0.125
400	800	800	0.125
800	200	200	0.125
800	400	400	0.125

The strategies are separated by separator lines. Thus, except for the bad luch first choice of 100, the expected reward of no-switching strategy is $\frac{1400}{4} = 350$₹ and switching strategy is $\frac{2800}{8} = 350$₹. Thus, no strategy is better than the other.