Birla Institute of Technology and Science, Pilani – K K Birla Goa Campus

2022-2023 Sem I	Mid-Sem Exam	03-11-2022
CS F222	Discr. Struc. for Comp. Sc.	09:30-11:00 hrs
Marks 70	Weightage 35%	Closed book, 90 min

Instructions: *Assumptions must be clear and justified. Consistency is a must, conformance not enough.*

$0 \in \mathbb{N}$ Rationals are $\mathbb{Q}$ and reals are $\mathbb{R}$.

Q1, basics Argue which of the following statements are clearly provable or refutable, and which are not, and justify why. Wherever possible, give **one line** proofs or disproofs each. Mixing formal and informal is okay for this purpose, please be brief and precise. **[28]**

(a) If for $a_0, n \in \mathbb{N}$, $a_0 \neq 0$, $a_{n+1} = 2a_n - \frac{a_0}{2}$, then for $n \in \mathbb{N},\ n > 0$, $\frac{a_n}{a_0} = 2^{n-1} + \frac{1}{2}$.
(b) The same as (a), except that $a_0 \in \mathbb{N}$ is not guaranteed, but $a_0 \neq 0$ remains true.
(c) If $a, b \in (0, 1) \cap \mathbb{Q}$, and we replace all digits 2-9 (except 0,1) by 0 in their decimal representations to obtain c, d respectively, then $a > b$ iff $c > d$.
(d) The same as (c), except that a, b are irrational.
(e) There are some $r, s \in \mathbb{Q}$ such that $3^r = 2^s$.
(f) There are some $r, s \in \mathbb{Q}$ such that $(1 + sqrt(3))^r = e^s$.
(g) The sum of decimal digits in the decimal representation of a prime number has to be prime.

Q2, induction Take the sequence $\langle a_n \rangle$ to be such that $a_0 = 0$ and $a_{n+1} = a_n + 2n + 1$. Answer the following.

(a) Give an enumeration (mapping from $\mathbb{N}$) to $\langle a_n \rangle$. In other words, express a_n as a function of n. **[3]**
Hint: First list out a few members of the sequence on a rough sheet.
(b) Using this sequence and this enumeration, give an induction proof that for $m, n \in \mathbb{N}$, $m > n$, $m^2 - n^2 = (m + n)(m - n)$. **[10]**
Hint: Take $m = n + k$. Do induction on k. Make use of the sequence given, not the well-known identities. That is, ***do not use the target result as an assumption****.*
(c) Show how one can generalize the result: that is, show how one can render the conditions $m, n \in \mathbb{N}$ and $m > n$ unnecessary. **[3]**

Q3, countability (a) Will the set $X = \{x^n : n \in \mathbb{N}, x \in \mathbb{R}, x > 0, x^2 - x - 11 = 0\}$ be countable? Justify. You may use known results to support the arguments, if and only if accurate references are given. Do not use vague allusions. **[10]**
Q3 (b) Given X as in above, will the set of real roots of polynomials with coefficients from $X \cup \mathbb{N}$ be countable? Justify. **[6]**

Q4, fresh Given the Fibonacci Sequence $F_0 = 1, F_1 = 1, F_{n+2} = F_n + F_{n+1}$, use induction only to prove one of the following. Derive the others drirectly from that. **[10]**
Prove: $\sum_{i=0}^{n} F_{2i} = F_{2n+1}$ and $\sum_{i=1}^{n} F_{2i-1} = F_{2n} - F_0$ and $\sum_{i=0}^{2n} F_i = F_{2n+2} - F_0$.

Solution

Q1, basics *To argue whether statements are clearly provable or refutable or not, a grip on elementary principles is necessary and sufficient.* **[28]**

(a) If for $a_0, n \in \mathbb{N}$, $a_0 \neq 0$, $a_{n+1} = 2a_n - \frac{a_0}{2}$, then for $n \in \mathbb{N},\ n > 0$, $\frac{a_n}{a_0} = 2^{n-1} + \frac{1}{2}$.
Answer: Well-formed statement, provable by induction. The base case is true, and $a_{n+1} = 2a_n - \frac{a_0}{2} \stackrel{I.H.}{=} 2 \times 2^{n-1}a_0 + a_0 - \frac{a_0}{2} = (2^n + \frac{1}{2})a_0$.

(b) The same as (a), except that $a_0 \in \mathbb{N}$ is not guaranteed, but $a_0 \neq 0$ remains true.
Answer: Nothing changes (as long as arithmetic on whatever domain a_0 belongs to is interpreted like the usual arithmetic on $\mathbb{N}$).*[Here: the main idea is that the induction, the sequence properties, depend on the form and the enumeration – the predecessor-successor **relationship**. **Not** on the absolute values and their domains.]*

(c) If $a, b \in (0,1) \cap \mathbb{Q}$, and we replace all digits 2-9 (except 0,1) by 0 in their decimal representations to obtain c, d respectively, then $a > b$ iff $c > d$.
Answer: Easily disprovable. (One counterexample is enough: e.g., $a = 0.1, b = 0.2$.)

(d) The same as (c), except that a, b are irrational.
Answer: Easily disprovable. (One counterexample is enough: e.g., $a = \frac{1}{\sqrt{3}}, b = \frac{1}{\sqrt{5}}$.)

(e) There are some $r, s \in \mathbb{Q}$ such that $3^r = 2^s$.
Answer: Easily proved by (the **unique**) example $r = s = 0$. If $r \neq s$ then the statement is false, because $\frac{\log 3}{\log 2} = \log_2 3$ is irrational.

(f) There are some $r, s \in \mathbb{Q}$ such that $(1 + sqrt(3))^r = e^s$. **Answer:** Easily proved by (the **unique**) example $r = s = 0$. If $r \neq s$ then the statement is either unknown or false, because $\log{(1 + \sqrt{3})}$ is irrational, and $\frac{s}{r}$ is rational.

(g) The sum of decimal digits in the decimal representation of a prime number has to be prime.
Answer: Easily disproved by an example such as 13.

Q2, induction $a_0 = 0$ and $a_{n+1} = a_n + 2n + 1$.

(a) Enumeration: $a_0 = 0, a_1 = 1, a_2 = 4, a_3 = 9, a_4 = 16,\ \ldots,\ \langle a_n \rangle = n^2$. **[3]**

(b) From the enumeration, $m^2 - n^2 = a_m - a_n$. For $m > n$, let $m = n + k, k > 0$. **[1]**
We are seeking to prove that $a_m - a_n = (2n + k)k$.
B.C.: $k = 1$, $m + n = 2n + 1$; $m - n = 1$; $a_m - a_n = 2n + 1 = (m + n)(m - n)$. **[1]**
The I.H. is that $a_{n+k} - a_n = (2n + k)k$ for some $k \geq 1$. **[2]**
$a_{n+k+1} - a_n = a_{n+k+1} - a_{n+k} + a_{n+k} - a_n$ **[1]**
$\stackrel{B.C.+I.H.}{=\!=\!=} 2(n + k) + 1 + (2n + k)k = 2n + 2k + 1 + 2nk + k^2 = (2n + k + 1)(k + 1)$: I.S. forded! **[5]**

(c) The question is not to generalise the result, but the **induction** proof. To generalise, for the sequence $\langle a_n \rangle : \mathbb{N} \to S$, the range S needs to have the elementary properties of arithmetic, like distributivity and associativity, as well as countability: the difference k must be countable to do induction on. Think of polynomials on a countable domain. Now the question that remains is, since the result is true for any domain that has

the elementary properties of distribution and associativity in arithmetic, why can't we prove it by the same induction over uncountable domains? **[2]**
The condition $m > n$ is indeed superfluous. It is not necessary for $\mathbb{N}$ also. It only needs associativity of unary and binary minus (i.e. $-(x-y) = y-x$). **[1]**

Q3, countability (a) The set $X = \{x^n : n \in \mathbb{N}, x \in \mathbb{R}, x > 0, x^2 - x - 11 = 0\}$ is countable, because $x = \frac{1+3\sqrt{5}}{2} = 3\varphi - 1$ where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio, and $X \subseteq Y \cup \mathbb{N}$. **[5]**
: Where $Y = \{\varphi^n : n \in \mathbb{N}\}$ is countable (it's a sequence; the definition itself is an enumeration). **[5]**
Independently, without alluding to φ also, this can be proved: x is a unique number, and X's elements are exactly the sequence members of $\langle x^n \rangle$. **[5+5=10]**
Q3 (b) Given X as in above, since X is countable, the set of real roots of polynomials with coefficients from $X \cup \mathbb{N}$ will be countable. Because the set of polynomials with coefficients from the countable union $X \cup \mathbb{N}$ will be countable (it has the same cardinality as the set of finite subsets of a countable set), and the number of roots of a polynomial are not more than the number of coefficients that determine the polynomial.
Conclusion 2 marks, $X \cup \mathbb{N}$ being countable 1 mark, the set of polynomials on countable being countable 1 mark, roots no more 1 mark.

Q4, fresh Given the Fibonacci Sequence $F_0 = 1, F_1 = 1, F_{n+2} = F_n + F_{n+1}$:
To Prove:

$$\sum_{i=0}^{n} F_{2i} = F_{2n+1} : \qquad \text{(Result 1)}$$

B.C. $\sum_{i=0}^{0} F_{2i} = F_0 = F_1$ (alternatively, $F_0 + F_2 = F_3$).

I.H. For some $k \geq 0$, $\sum_{i=0}^{k} F_{2i} = F_{2k+1}$.

I.S. $\sum_{i=0}^{k+1} F_{2i} = \sum_{i=0}^{k} F_{2i} + F_{2k+2}$
$\overset{I.H.}{=} F_{2k+1} + F_{2k+2}$
$= F_{2k+3} = F_{(2(k+1)+1)}$ $\square$.

Direct derivations: Since $F_{2i-1} = F_{2i} - F_{2i-2}$, we get that

$$\sum_{i=1}^{n} F_{2i-1} = \sum_{i=1}^{n} \left(F_{2i} - F_{2(i-1)}\right) = \sum_{i=0}^{n} F_{2i} - \sum_{i=0}^{n-1} F_{2i} - F_0.$$

Using (Result 1) proven above,

$$\sum_{i=1}^{n} F_{2i-1} = F_{2n+1} - F_{2n-1} - F_0 = F_{2i} + F_{2i-1} - F_{2i} = F_{2n} - F_0$$

Combining the two,

$$\sum_{i=0}^{2n} F_i = \sum_{i=0}^{n} F_{2i} + \sum_{i=1}^{n} F_{2i-1} = F_{2n+1} + F_{2n} - F_0 = F_{2n+2} - F_0.$$

Alternatives

To prove $\sum_{i=1}^{n} F_{2i-1} = F_{2n} - F_0$:

B.C. $F_1 = F_2 - F_0$.

I.H. $\sum_{i=1}^{k} F_{2i-1} = F_{2k} - F_0$.

I.S. $\sum_{i=1}^{k+1} F_{2i-1} = \sum_{i=1}^{k} F_{2i-1} + F_{2k+1}$
$\overset{I.H.}{=} F_{2k} + F_{2k+1} - F_0$
$= F_{2k+2} - F_0$. $\square$

To prove $\sum_{i=0}^{2n} F_i = F_{2n+2} - F_0$:

B.C. $F_0 = F_2 - F_0$.

I.H. $\sum_{i=0}^{2k} F_i = F_{2k+2} - F_0$.

I.S. $\sum_{i=0}^{2(k+1)} F_i = \sum_{i=0}^{2k} F_i + F_{2k+2}$
$\overset{I.H.}{=} F_{2k} + F_{2k+1} - F_0$
$= F_{2k+2} - F_0$. $\square$

For both these cases, we show only $\sum_{i=0}^{2n} F_i = F_{2n+2} - F_0 \Rightarrow \sum_{i=0}^{n} F_{2i} = F_{2n+1}$ as the rest is easy to pick up and connect from the first case.
$\sum_{i=0}^{2n} F_i = \sum_{i=0}^{n-1} (F_{2i} + F_{2i+1}) + F_{2n} = \sum_{i=0}^{n-1} F_{2i+2} + F_{2n} = \sum_{i=0}^{n} F_{2i} + F_{2n} - F_0$, thus,
$F_{2n+2} - F_0 = \sum_{i=0}^{n} F_{2i} + F_{2n} - F_0 \Rightarrow \sum_{i=0}^{n} F_{2i} = F_{2n+2} - F_{2n} = F_{2n+1}$ $\square$