Design and Analysis of Algorithms (CS F364), **Mid-Semester Test**
Second Semester 2023 **56 marks**

Instructions

a All questions are compulsory.

b You can use propositions that were proved in class, i.e., without re-proving them.

c Be precise.

d This is an open book exam. You can use printed or handwritten material.

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1. **(3 marks)** Suppose to solve a problem P, you are choosing between the following three algorithms proposed by:

   a). **Friend who is coming to the class regular:** Algorithm solves problems by dividing them into five subproblems of half the size, recursively solving each subproblem, and then combining the solutions in linear time. Assume that T(1) = 1.

   b). **Friend who is not coming to the class regular:** Algorithm solves problems of size n by recursively solving two subproblems of size $n-1$ and then combining the solutions in constant time. Assume that T(1) = 1.

   c). **ChatGpt:** Algorithm solves problems of size n by dividing them into nine subproblems of size $\frac{n}{3}$, recursively solving each subproblem, and then combining the solutions in $O(n^2)$ time. Assume that T(1) = 1.

   Who is giving you the best algorithm in terms of time complexity? Justify the reason.

2. **(3 marks)** To find the k-th smallest element in an array of n elements, suppose that we modify the Select algorithm, that is, the Median of Median algorithm, by breaking the elements into groups of size m, where m is an odd number. Write a recurrence to analyze the run-time of the algorithm if we choose to break the elements into groups of size m ?

3. **(5 marks)** Prove using induction that $T(n) = O(\log n)$ for the below recurrence.

   $$T(1) = c.$$

   $$T(n) = T(n/2) + d, n > 1.$$

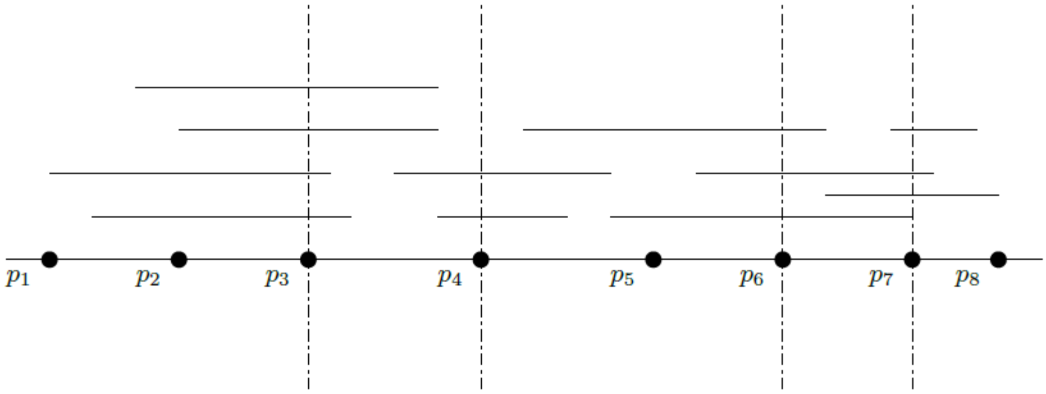
   Here c and d are constants.

4. **(5 marks)** Given a directed graph $G = (V, E)$, the start node s and the destination node t, let A be a path from s to t and B be another path from s to t, both paths are vertex-disjoint paths if they do not contain a common vertex (except s and t). Find the maximum number of vertex-disjoint paths from s to t.

5. **(10 marks)** Let A be an $n \times n$ matrix of integers such that each row and each column are arranged in ascending order. We want to check whether a number k appears in A. If k is present, we should report its position, that is, the row i and the column j such that $A(i, j) = k$. Otherwise, we should declare that k is not present in A. Describe an algorithm that solves this problem in linear time. Justify the complexity of your algorithm. For your algorithm, describe a worst-case input where k is present in A.

6. **(10 marks)** You are given a set $P = \{p_1, p_2, \ldots, p_n\}$ of n points on a line. Note that p_i is left of p_j, where $1 \le i < j \le n$. You are also given a set of m intervals $I_1, I_2, \ldots, I_m$, where each interval I_j is of the form $[start_j, end_j]$ and $p_1 \le start_j \le end_j \le p_n$. Design an algorithm to find a subset $X \subseteq P$ of the smallest cardinality such that each interval contains at least one point from X. Justify the correctness.

   For example, in the figure mentioned below, $X = \{p_3, p_4, p_6, p_7\}$ is a subset of 4 points such that each interval contains at least one point from the subset X.



7. **(10 marks)** I got bored with the current tutorial process. Next semester, I plan to give N quizzes (instead of tutorials), with negative marking. Furthermore, I am even bored of the usual "Best M out of N tutorials" formula to award marks for internal assessment. Instead, each student in the DAA course will be evaluated based on the sum of the best contiguous segment (i.e., no gaps) of marks in the overall sequence of quizzes. However, the student is allowed to drop up to K quizzes before calculating this sum. Suppose that a student has scored the following marks in 10 quizzes during the semester.

Quiz	1	2	3	4	5	6	7	8	9	10
Marks	6	−5	3	−7	6	−1	10	−8	−8	8

   Without dropping any quizzes, the best segment is the quiz $5-7$, which yields a total of 15 marks. If the student is allowed to drop up to 2 quizzes in a segment, the best segment is the quiz $1-7$, which yields a total of 24 marks after dropping quizzes 2 and 4. If the student is allowed to drop up to 6 quizzes in a segment, the best total is obtained by taking the entire list and dropping all 5 negative entries, yielding 33 marks.

   For $1 \le i \le N$, $0 \le j \le K$, let $B[i][j]$ denote the maximum sum segment ending at position i with at most j marks dropped.

   (a) Write a recursive formula for $B[i][j]$.

   (b) Explain how to calculate, using dynamic programming (basically, explain how to fill the table).

   (c) Describe the time complexity of your dynamic programming algorithm.

8. **(10 marks)** Show that for any bipartite graph $G = (V, E)$, the maximum size of a matching is exactly equal to the size of the minimum vertex cover. *Recall that a subset of vertices $S \subseteq V$ is a vertex cover if every edge in E has at least one endpoint in S.*