Design and Analysis of Algorithms (CS F364) Comprehensive Examination (Second Semester, 2023)

There are 6 questions in all and total marks are $12 + 12 + 16 + 16 + 16 + 16 = 88$. The questions are arranged in increasing order of difficulty. Please show all steps in computations or proofs with complete derivation of complexity results. This is an **open book exam**. You can use any printed or handwritten material restricted to the text book, reference books, and lecture notes. Calculators are allowed. Time: 180 minutes.

1. Using extended Euclid's algorithm for GCD find the gcd $(677801, 939871)$, and also integers x and y such that $(677801, 939871) = 677801x + 939871y$, showing all the steps involved.

2. By making use of efficient algorithms, find the last three (least significant) digits in the decimal expansion of

$$20232023202320232023^{20232023202320232023}.$$

3. Design an algorithm that on input $n \in \mathbb{N}$ computes the decimal representation of 2^n in $O(n^{\log_2 3})$ time. (*Decimal representation examples:* $2^1 = 2, 2^5 = 32, 2^0 = 1$.)

4. Consider the following factor 2 approximation algorithm for the cardinality vertex cover problem: Find a depth first search tree in the given graph G, and output the set, say S, of all the nonleaf vertices of this tree. Prove that S is indeed a vertex cover for G and

$$|S| \leq 2 \cdot \text{OPT},$$

where OPT is the size of the minimum cardinality vertex cover of G.

5. Let D_n be a probability distribution over $\{0,1\}^n$. D_n is called $\mathbf{P}$-*computable* if there exists a polynomial-time DTM that, given input $x \in \{0,1\}^n$, can compute the *cumulative probability* $\mu_{D_n}(x)$, where

$$\mu_{D_n}(x) = \sum_{y \in \{0,1\}^n | y \leq x} \Pr_{D_n}[y].$$

Here $\Pr_{D_n}[y]$ denotes the probability assigned to string y according to the probability distribution D_n and $y \leq x$ means y either precedes x in lexicographic order or is equal to x.

Let $G_{n,p}$ be the probability distribution over n-vertex graphs ($n \times n$ binary adjacency matrices) where each edge is chosen to appear in the graph independently with probability p. Prove that $G_{n,p}$ is $\mathbf{P}$-*computable.*

6. For an edge-weighted undirected graph $G = (V, E, w)$, let the weight function be $w : E \to \mathbb{N}$ and let the weight bound be $W \in \mathbb{N}$. We define the language `MAX-CUT` as follows:

`MAX-CUT`

$$= \{ ((V, E, w), W) \mid \exists \text{ a set } S \subseteq V \text{ such that } \sum_{\{u,v\} \in E, u \in S, v \notin S} w(u,v) \geq W \}.$$

Prove that `MAX-CUT` is **NP-COMPLETE**.