Advanced Algorithms & Complexity (CS G526) Mid Sem Exam, 2022

There are 5 questions in all and total marks are $5 \times 5 = 25$. This is an open book exam. You can use any printed or handwritten material. Calculators are not allowed. Please show all steps of your solution and give full derivation of your results.

1. We have defined a relation $\leq_p$ among languages. We noted that it is *reflexive* (i.e., $L \leq_p L$ for all languages L) and *transitive* (i.e., if $L \leq_p L'$ and $L' \leq_p L''$ then $L \leq_p L''$). Prove that it is not *symmetric*, namely, $L \leq_p L'$ need not imply $L' \leq_p L$.

2. We define a language L as follows:

   $$L = \{(M, 1^t) \mid \forall x \text{ such that } |x| \leq t, \text{ DTM } M \text{ accepts } x \text{ within } t \text{ steps}\}$$

   Notation: 1^t is 1 written t times, where t is an integer. $|x|$ is the length the string x.
   Prove or disprove:
   $$L \in \texttt{PSPACE}.$$

3. Prove that the following is an alternative definition of `BPP`:
   A Language L is in `BPP` if there exists a polynomial-time DTM M and a polynomial $p : \mathbb{N} \to \mathbb{N}$ such that for every $x \in \{0,1\}^*$, $\Pr_{r \in_R \{0,1\}^{p(|x|)}}[M(x,r) = L(x)] \geq \frac{2}{3}$.
   Notation: $\in_R$ means randomly chosen. $L(x) = 1$ if $x \in L$ and $L(x) = 0$ if $x \notin L$.

4. Assuming that the two players R (the *row player*) and C (the *column player*) are using mixed strategies, determine the values V_R and V_C of the following 2×2 matrix game, and give optimal mixed strategies for the two players:

   $$\begin{pmatrix} 4 & 9 \\ 7 & 2 \end{pmatrix}.$$

5. Using *Euclid's Extended GCD Algorithm*, solve the following congruence:

   $$249x + 267 \equiv 0 (\text{mod } 291).$$