Advanced Algorithms & Complexity (CS G526) Mid Sem Exam, 2023

There are 5 questions in all and total marks are $(1+1+3)+(2+3)+(5)+(5)+(2+3) = 25$. *This is an open book exam. You can use any printed or handwritten material. Calculators are allowed. Please show all steps of your solution and give full derivation of your results using efficient algorithms.*

1. (a) In the *SUBSET SUM* problem we are given a list of n numbers $A_1, \ldots, A_n$ and a number T and need to decide whether there exists a subset $S \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in S} A_i = T$ (the language L_1). Prove or disprove

$$L_1 \in \texttt{NP}.$$

   (b) We formulate the *Load Balancing* problem as follows. We are given a set of m machines $M_1, \ldots, M_m$ and a set of n jobs; each job j has a processing time t_j. We seek to assign each job to one of the machines so that the loads placed on all machines are as "balanced" as possible. More concretely, in any assignment of jobs to machines, we can let $A(i)$ denote the set of jobs assigned to machine M_i; under this assignment, machine M_i needs to work for a total time of $T_i = \Sigma_{j \in A(i)} t_j$, and we declare this to be the *load* on machine M_i. We seek to minimize a quantity known as the *makespan*; it is simply the maximum load on any machine, $T = \max_i T_i$. Give a decision version of the *Load Balancing* problem (the language L_2).

   (c) Prove or disprove

$$L_1 \leq_p L_2.$$

2. We define a language L as follows:

$$L = \{ (M, x) \mid \text{ TM } M \text{ accepts } x \}.$$

   (a) Prove or disprove:

$$L \in \texttt{PSPACE-COMPLETE}.$$

   (b) Prove or disprove:

$$L \in \texttt{PSPACE-HARD}.$$

3. Let $f(x_1, x_2, x_3, x_4) = (\neg x_1 \vee \neg x_3) \wedge (\neg x_2 \wedge \neg x_4)$. Two players P_1 and P_2 are playing the QBF game with $f(x_1, x_2, x_3, x_4)$ as the Boolean function. P_1 first selects the value of x_1, then P_2 selects the value of x_2, then P_1 selects the value of x_3, then P_2 selects the value of x_4. Finally $f(x_1, x_2, x_3, x_4)$ is evaluated. P_1 wins if $f(x_1, x_2, x_3, x_4)$ evaluates to 1, otherwise P_2 wins. Draw the game tree corresponding to this QBF game and evaluate it. Find whether P_1 has a winning strategy or not. If P_1 is having a winning strategy, then describe it.

4. Show the working of the work-optimal EREW PRAM algorithm for adding n integers on the following input:

$$1, 17, 2, 18, 3, 19, 4, 20, 5, 21, 6, 22, 7, 23, 8, 24$$
$$9, 32, 10, 31, 11, 30, 12, 29, 13, 28, 14, 27, 15, 26, 16, 25.$$

5. (a) Find the GCD, $(36, 144, 200)$, and the integers x, y, z such that

$$(36, 144, 200) = 36x + 144y + 200z.$$

   (b) Prove that there exist integers $x_1, \ldots, x_n$ such that the GCD of n integers, $(a_1, \ldots, a_n)$, can be represented as their linear combination:

$$(a_1, \ldots, a_n) = \sum_{j=1}^{n} a_j x_j.$$