Mid-Semester Test

Instructor-in-Charge: B. Sainath, 2210−A, Electrical and Electronics Eng. Dept., BITS Pilani.

Course No./Title : EEE F311/Communication Systems DATE: 11th Oct. 2017

Max. marks: 50 Test duration: 90 Mins.

Important instructions: Exam is fully closed book. Give all your answers with proper units. In all sketches, include x-label, y-label, and, figure caption. Incomplete figures will not get full credit.

Part-I: Multiple choice questions (10 marks)

Note: Each question carries two marks. Write the right answer in your answer book. For each incorrect answer, you lose **0.5 mark**. Overwritten answers will **NOT** be evaluated.

1. Let $\hat{m}(t)$ denote the Hilbert transform of $m(t)$. If $m(t) = \frac{1}{1+t^2}$, $\hat{m}(1)$ is equal to

A. $\frac{1}{3}$. B. $\frac{1}{2}$. C. $\frac{1}{4}$ D. zero

2. Let R be a Rayleigh distributed random variable, defined by its probability density function (PDF)

$$p_R(r) = \begin{cases} re^{-\frac{r^2}{2}}, & r \geq 0, \\ 0, & otherwise \end{cases}.$$

The complementary cumulative distribution function (CCDF) value at $r = \sqrt{2}$ is equal to

A. $\frac{1}{e}$ B. $1 - \frac{1}{e}$ C. $\frac{1}{e^2}$ D. $1 - \frac{1}{e^2}$

3. A modulated signal is given by $s(t) = [A_c + m(t)]\cos(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t)$, where $m(t)$ denotes the modulating signal and $M(f)$ denotes its Fourier transform. If $s(t)$ is applied as input to an envelop detector, the spectrum of approximate output when $A_c >> |m(t)|, A_c >> |\hat{m}(t)|$ for all t, is given by

A. $A_c + M(f)$ B. $A_c\delta(f) + M(f)$ C. $A_c\delta(f)M(f)$ D. $A_cM(f)$

4. An FM wave having a frequency deviation of 9.0 KHz at a modulation frequency of 3.6 KHz is applied to a cascaded combination of a frequency doubler and a tripler. The frequency deviation, and the modulation index of the FM signal at the output, respectively, are

A. 54 KHz, 7.5 B. 108 KHz, 7.5 C. 54 KHz, 15 D. 108 KHz, 15

5. Suppose that an AM broadcast station transmits a total power of 50 kilowatt when the RF carrier is modulated by a sinusoidal message signal with modulation index $\frac{1}{\sqrt{2}}$. Assume that the antenna represents a load of 50 Ω. Then, the transmission efficiency, and the carrier's peak amplitude, respectively, are

A. 20%, 0.2 KV B. 20%, 2 KV C. 10%, 0.2 KV D. 10%, 20 KV

Part II: Answer the following. Show key steps. Highlight final answers in rectangular boxes. Avoid overwriting. Use 'j' to represent $\sqrt{-1}$. Simplify your answers to the extent possible.

Q. 1. Suppose that, for a given excitation $x(t)$ to a system, output is given by

$$y(t) = \int_{t-1}^{t} x(\tau)\, d\tau.$$

Let $X(f)$ denote the Fourier transform of $x(t)$.

i). Determine the Fourier transform of $y(t)$, denoted by $Y(f)$. [3 marks]

ii). Determine the transfer function $H(f)$ of the system. Compute H(f) value at $f = 0$. [2 marks]

iii). Write down the expressions for following:

a). Amplitude response $|H(f)|$. b). Phase response $\angle H(f)$. c). Impulse response of the system. [3 marks]

iv). Determine the output spectral density at $f = \frac{1}{2\pi}$ when $x(t) = e^{-t}u(t)$, where $u(t)$ denotes the unit-step function. Simplify your answer to a positive real number. [2 marks]

Q. 2. Consider PDF of a continuous random variable (CRV) Γ, which is given by

$$p_\Gamma(\gamma) = \frac{\epsilon}{\sqrt{2\pi}\sigma\gamma} \exp\left(-\frac{(10\log_{10}\gamma - \mu)^2}{2\sigma^2}\right), \gamma > 0,$$

where $\epsilon = \frac{10}{\ln 10}$, and μ (dB) and σ^2 (dB) are the mean, and the variance of $10\log_{10}\Gamma$, respectively. Note that $10\log_{10}\Gamma$ has Gaussian distribution.

i). Derive an expression of $\mathbb{F}_\sigma$, which is given by

$$\mathbb{F}_\sigma = \frac{\mathbf{E}\left[\Gamma^2\right] - \left(\mathbf{E}\left[\Gamma\right]\right)^2}{\left(\mathbf{E}\left[\Gamma\right]\right)^2},$$

where $\mathbf{E}[\Gamma]$ denotes the mean value of the CRV Γ, and $\mathbf{E}\left[\Gamma^2\right]$ denotes the mean square value of Γ. [3 + 2 + 1 marks]

ii). Sketch the derived expression of $\mathbb{F}_\sigma$ for 0 dB $\leq \sigma^2 \leq$ 20 dB. [1.5 marks]

Q. 3. Suppose that, a wide-band phase modulated (PM) signal is obtained by using a single-tone modulating signal $m(t) = A_m \cos(2\pi f_m t)$. Assume that the modulator phase sensitivity is k_p radians/volt. Furthermore, β_p denotes the phase modulation index and let f_c denote the RF carrier frequency.

i). Derive an expression for the instantaneous phase of the PM signal, denoted by $\theta_i(t)$. [2 marks]

ii). Obtain an expression for the instantaneous frequency of the PM signal, denote by $f_i(t)$. [1 mark]

iii). Using Carson's rule, derive an expression for the transmission bandwidth B_T. For $\beta_p >> 1$, write down the approximate expression. [2 + 1 marks]

iv). In wide-band FM, the modulation index $\beta_f >> 1$, what is the approximate transmission bandwidth? Compare and comment on B_T of wide-band PM signal and wide-band FM signal. [1.5 marks]

Q. 4. Consider a bandpass signal $g(t)$ of duration T. The spectrum of $g(t)$ lies in the interval $f_c - W \leq |f| \leq f_c + W$, where f_c is the mid-band frequency of $g(t)$.

i). Write down the canonical representation of $g(t)$. [1 mark]

ii). Draw a scheme to derive the in-phase and quadrature components of $g(t)$. [2 marks]

iii). Prove or disprove:

"The bandpass signal $g(t)$ is uniquely described by specifying $2WT$ samples of its in-phase and $2WT$ samples of its quadrature components." [3 marks]

iv). Suppose that, the bandpass signal has duration of 0.6 millisecond. If $f_c = 900$ MHz and $W = 12.5$ MHz, compute the number of samples in the bandpass signal. [1 mark]

Q. 5. Consider an SSB signal which is obtained by modulating a carrier $c(t) = A_c \cos(2\pi f_c t)$ using a lowpass signal $m(t)$. The Hilbert transform of $m(t)$ is denoted by $\hat{m}(t)$.

i). Write down the time-domain representation of the SSB signal. [1 mark]

ii). Determine the envelope of the upper sideband (USB) signal and the lower sideband (LSB) signal. [1 mark]

iii). Determine the instantaneous phase and frequency of the SSB signal when a). only USB is transmitted. b). only the LSB is transmitted. [2 + 2 marks]

iv). Suppose that $m(t) = \cos(2\pi f_m t)$. Determine the envelope and instantaneous frequency when a). only USB is transmitted. b). only the LSB is transmitted. [2 marks]

□ END OF QUESTION PAPER □

Answers

Part I

1. B. Show that $\hat{m}(t) = \frac{t}{1+t^2}$. So, $\hat{m}(1) = \frac{1}{2}$.

2. A. The complementary CDF $F_R^c(r) = e^{-\frac{r^2}{2}}, r \geq 0$. So, $F_R^c(\sqrt{2}) = \frac{1}{e}$.

3. B. Output of envelope detector is approximately $A_c + m(t)$. So, the spectrum is given by $A_c\delta(f) + M(f)$.

4. C. O/P signal Frequency deviation = $9 \times 6 = 54$ KHz. So, the modulation index $\beta = \frac{54}{3.6} = 15$.

5. B. Transmission efficiency = $\frac{\mu^2}{2+\mu^2} = 20\%$. Since carrier power $\frac{{A_c}^2}{2R_L} = 40$ KW, the carrier amplitude = 2 KV.

Part II

1. Ans. i). The integrator output is

$$y(t) = \int_t^{t-1} \int_{-\infty}^{\infty} X(f) e^{j2\pi f\tau}\, d\tau,$$
$$= \int_{-\infty}^{\infty} X(f)\, \mathrm{sinc}(f)\, e^{-j\pi f} e^{j2\pi f t}\, df.$$

Therefore, the Fourier transform of the integrator output $y(t)$ is given by

$$Y(f) = X(f)\, \mathrm{sinc}(f)\, e^{-j\pi f}.$$

ii). It is obvious that $y(t)$ can be obtained by passing the excitation $x(t)$ through a filter whose transfer function is equal to

$$H(f) = \mathrm{sinc}(f)\, e^{-j\pi f}.$$

$H(f)$ when $f = 0$ is equal to 1 since $\mathrm{sinc}(0) = 1$, defined as the limit.

iii). a). Amplitude response $|H(f)| = \mathrm{sinc}(f)$, for all f.

b). Phase response $\angle H(f) = -\pi f \pm n\pi$, for all f, where n is an integer including zero.

c). Impulse response $h(t)$ is the inverse Fourier transform of $H(f)$. So, $h(t) = \mathrm{rect}\left(t - \frac{1}{2}\right)$.

iv). Since $X(f) = \frac{1}{1+j2\pi f}$, the output spectral density $\mathrm{S}_Y(f)$ in terms of the input spectral density $\mathrm{S}_X(f) = |H(f)|^2$ is given by

$$\mathrm{S}_Y(f) = |H(f)|^2\, \mathrm{S}_X(f),$$
$$= \mathrm{sinc}^2(f) \times \frac{1}{1 + 4\pi^2 f^2}.$$

At $f = \frac{1}{2\pi}$, $\mathrm{S}_Y(f)$ is equal to $\mathrm{sinc}^2(\frac{1}{2\pi}) \times 0.5 = 0.4597$.

2. Ans. i). First, determine the mean value (MV).

MV: Let $Y = 10\log_{10}\Gamma \Rightarrow \Gamma = 10^{\frac{Y}{10}}$. Note that Y has Gaussian distribution. The MV is given by

$$\mathbf{E}[\Gamma] = \mathbf{E}\left[10^{\frac{Y}{10}}\right],$$
$$= \mathbf{E}\left[e^{\log_e 10^{\frac{Y}{10}}}\right],$$
$$= \mathbf{E}\left[e^{\frac{Y}{\epsilon}}\right],$$

where $Y \sim \mathcal{N}(\mu, \sigma^2)$ and $\epsilon = \frac{10}{\log_e 10}$. Using $\mathbf{E}\left[e^{tY}\right] = e^{\mu t + \frac{\sigma^2 t^2}{2}}$,

$$\mathbf{E}[\Gamma] = e^{\frac{\mu}{\epsilon} + \frac{\sigma^2}{2\epsilon^2}}.$$

MSV: The MSV is given by

$$\mathbf{E}\left[\Gamma^2\right] = \mathbf{E}\left[e^{\frac{2Y}{\epsilon}}\right],$$
$$= e^{\frac{2\mu}{\epsilon} + \frac{2\sigma^2}{\epsilon^2}}.$$

Using the MV and the MSV of Γ, we can show that

$$\mathbb{F}_\sigma = \exp\left(\frac{\sigma^2}{\epsilon^2}\right) - 1.$$

ii). $\mathbb{F}_\sigma$ as a function of σ^2 (dB) is plotted in Figure 1.

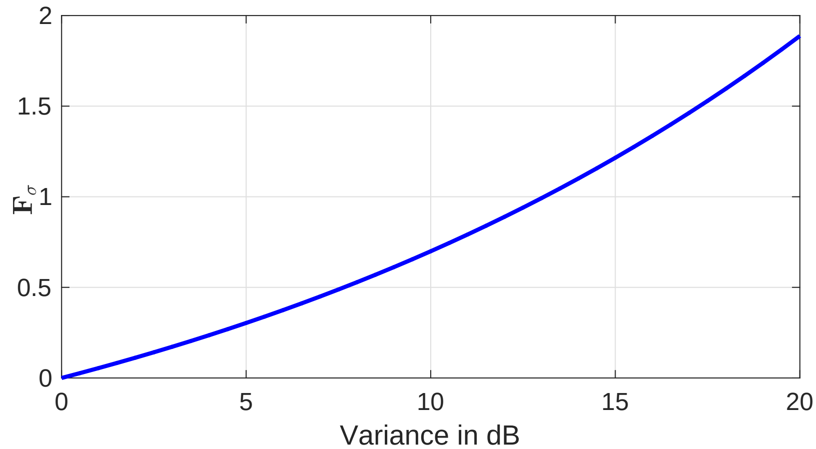

Fig. 1: $\mathbb{F}_\sigma$ as a function of variance.

3. ans. i). The instantaneous phase of the PM signal is given by

$$\begin{aligned}\theta_i(t) &= 2\pi f_c t + k_p m(t), \\ &= 2\pi f_c t + \beta_p \cos\left(2\pi f_m t\right),\end{aligned}$$

where $\beta_p = k_p A_m$. So, the instantaneous frequency of the PM signal is given by

$$\begin{aligned}f_i(t) &= \frac{1}{2\pi}\frac{d\theta_i(t)}{dt}, \\ &= f_c - \beta_p f_m \sin\left(2\pi f_m t\right).\end{aligned}$$

Observe that the maximum frequency deviation in a PM signal varies linearly with f_m.

iii). Using Caron's rule, the transmission bandwidth is given by $B_T = 2(\beta_p f_m + f_m) = 2f_m(1 + \beta_p)$.

For $\beta_p >> 1$, the transmission bandwidth is approximately given by $B_T \approx 2f_m\beta_p$.

iv). $\beta_f >> 1$, the transmission bandwidth of FM signal is approximately given by $B_T \approx 2\Delta f$. In this scenario, B_T is effectively independent of f_m. On the other hand, approximate B_T of PM signal varies linearly with f_m.

4. ans. i). The canonical representation of $g(t)$ in terms of its in-phase component $g_c(t)$ and quadrature component is given by

$$g(t) = g_c(t)\cos\left(2\pi f_c t\right) - g_s(t)\sin\left(2\pi f_c t\right).$$

ii). The scheme to derive $g_c(t)$ and $g_s(t)$ is shown below.

iii). Note that if the bandpass signal $g(t)$ has a bandwidth $2W$, centered at the carrier frequency f_c, then both $g_c(t)$ and $g_s(t)$ are lowpass signals, each with a bandwidth equal to W. So, sampling $g_c(t)$ and $g_s(t)$ at their Nyquist rate of 2W yields 2WT samples for each component in the time duration T. These samples uniquely define the lowpass signals $g_c(t)$ and $g_s(t)$. Since we can recover the original bandpass signal $g(t)$ from $g_c(t)$ and $g_s(t)$, then these two sets of samples also uniquely define $g(t)$.

iv). Number of samples = $2WT = 25 \times 10^6 \times 0.6 \times 10^{-3} = 15000$ samples.

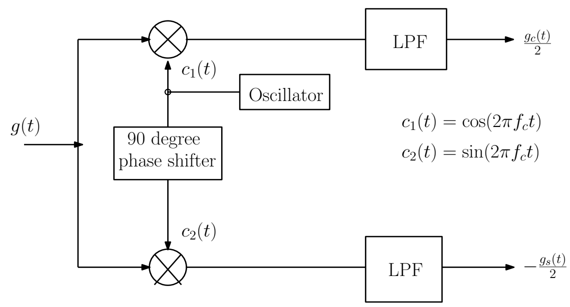

Fig. 2: Scheme for deriving in-phase and quadrature components.

5. ans. i).The SSB signal is given by

$$s(t) = \frac{A_c}{2}\left(m(t)\cos\left(2\pi f_c t\right) \pm \hat{m}(t)\sin\left(2\pi f_c t\right)\right).$$

ii). For both USB and LSB, the envelope is equal to $\frac{A_c}{2}\sqrt{m^2(t) + \hat{m}^2(t)}$.

iii). a). *USB:* The instantaneous phase is given by

$$\theta_i(t) = 2\pi f_c t + \tan^{-1}\left(\frac{\hat{m}(t)}{m(t)}\right).$$

The instantaneous frequency is given by

$$\begin{aligned} f_i(t) &= \frac{1}{2\pi}\frac{d\theta_i(t)}{dt}, \\ &= f_c + \frac{1}{2\pi}\frac{m(t)\hat{m}'(t) - \hat{m}(t)m'(t)}{m^2(t) + \hat{m}^2(t)}. \end{aligned}$$

where $'$ denotes derivative with respect to time.

b). *LSB:* The instantaneous phase is given by

$$\theta_i(t) = 2\pi f_c t + \tan^{-1}\left(-\frac{\hat{m}(t)}{m(t)}\right).$$

The instantaneous frequency is given by

$$f_i(t) = f_c + \frac{1}{2\pi}\frac{\hat{m}(t)m'(t) - m(t)\hat{m}'(t)}{m^2(t) + \hat{m}^2(t)}.$$

iv). a). Envelope is equal to $\frac{A_c}{2}$.

b). For USB transmission, the instantaneous frequency is $f_c + f_m$. For LSB transmission, the instantaneous frequency is $f_c - f_m$.