COMPREHENSIVE EXAMINATION

Instructor-in-charge: B. Sainath, 2210−A, Electrical and Electronics Eng. Dept., BITS Pilani.

Course No./Title : EEE G613/Advanced Digital Signal Processing

DATE: Dec. 17th, 2022. Test duration: 180 Mins. Max. marks: 40.

(*Note*: You may use standard results or formulae by stating them explicitly. Highlight final answers in boxes. Simplify your response to the extent possible. Use notation consistently. Provide key intermediate steps.)

Q.1. Consider the following filters:

$$\mathcal{H}_0(z) = 1 + z^{-1},\ \mathcal{H}_1(z) = 1 - z^{-1},\ \mathcal{F}_0(z) = 1 + z^{-1},\ \mathcal{F}_1(z) = -1 + z^{-1},$$

where $\mathcal{H}_0(z)$ and $\mathcal{H}_1(z)$ are the analysis filters, $\mathcal{F}_0(z)$ and $\mathcal{F}_1(z)$ are the synthesis filters.

Can we construct two-channel QMF bank using the above set of filters? Yes/No? Justify your response. [1 point]

ii). Suppose the synthesis filters are selected such that $\mathcal{F}_0(z) = \mathcal{H}_0(z)$ and $\mathcal{F}_1(z) = -\mathcal{H}_1(z)$. Comment on the nature of the matrix. Determine the Alias component matrix and its determinant. [3 points]

iii). Neatly sketch the QMF bank with analysis and synthesis filters. Provide all labels/notation in the block diagram. [2 points]

iv). Express the output $\widehat{X}(z)$ as a product of three matrices in which the middle matrix is the Alias component matrix. [1 point]

Q.2. Let $\sigma_0^2, \sigma_1^2, \ldots, \sigma_{N-1}^2$ denote the subband variances. Recall that variance signifies the power of a signal. Consider the coding gain in an ideal subband coding system with N subband channels (ideal filters). The subband coding gain (say $\mathcal{G}_c$) is defined as

$$\mathcal{G}_c = \frac{\frac{1}{N}\sum_{m=0}^{N-1}\sigma_m^2}{\left(\Pi_{m=0}^{N-1}\sigma_m^2\right)^{\frac{1}{N}}}.$$

Consider the following power spectrum of the input signal:

$$\left|X(e^{j\omega})\right|^2 = 1 - \frac{\omega}{\pi},\ \ |\omega| \le \pi.$$

i). Let $N = 2$. Determine subband variances (σ_0^2, σ_1^2) and their geometric mean (GM). Compute the subband gain in dB. *Note:* During the computation of variances, you need to account for positive and negative frequencies. [3 points]

ii). Consider the general scenario of N subband channels. Obtain the general expression for subband variance (say σ_k^2) and subband coding gain (as a function of N). *Note:* Final answer should be in the most simplified form and does *not* contain summation ('$\sum$').

[4 points]

Q.3. i). Consider the discrete-time (DT) matched filter detector. Consider the following statement: "The signal output of the matched filter is maximized by sampling the output at $n = (N-1)$." Prove this statement analytically. The following equation is useful:

$$y[n] = \int_{-0.5}^{0.5} S^*(f)X(f)\exp\{j2\pi f\,(n-(N-1))\}\,df.$$

ii). Consider the following signal detection:

$$s[n] = A_m\cos(2\pi f_0 n),\ 0 < f_0 < 0.5,\ n = 0, 1, \ldots, (N-1).$$

a). Determine the signal output of a discrete-time matched filter at time $n = (N-1)$. Simplify to the extent possible. *Hint:* Try approximation (sum → integral) for large N. *Note:* Your final answer should *not* contain summation ('$\sum$').

b). Assume that the signal is delayed by $n_0 > 0$ samples so that we receive $s[n-n_0]$. Applying the same matched filter (in part (a)), determine the signal output as a function of N, A_m, f_0, n_0 at $n = (N-1)$. Comment on the final result (i.e., What is the impact of n_0?) *Note:* Your final answer should *not* contain summation ('$\sum$'). [2 + 2 + 3 points]

[P.T.O]

Q.4. Consider the following real-valued measurements described by

$$Y_1 = \beta + W_1, \quad Y_2 = \beta + W_2,$$

where β is a non-random parameter, W_1, W_2, are random, statistically independent unbiased measurement errors having mean zero. The objective is to design a data processing algorithm that combines the two measurements Y_1 and Y_2 to produce mean square error (MSE)-optimal estimate of β. The errors W_1, W_2 have variances σ_1^2 and σ_2^2, respectively.

Define $\widehat{\beta} = k_1 Y_1 + k_2 Y_2$. Determine the optimal estimate $\widehat{\beta}^*$ in terms of the error variances and measurements.

Note: Let $\beta_e = \widehat{\beta} - \beta$. $\mathbf{E}[\beta_e] = 0$. Find k_1 and k_2 (in terms of noise variances) that minimize $\mathbf{E}\left[|\beta_e|^2\right]$.

Determine the minimum mean square estimation error (MMSE). [4 + 1.5 points]

ii). Refer to Fig. 1. Identify and name the signal processing tasks. Note that (a), (b), and (c) illustrate three types of estimation problems; we desire an estimate at time t.

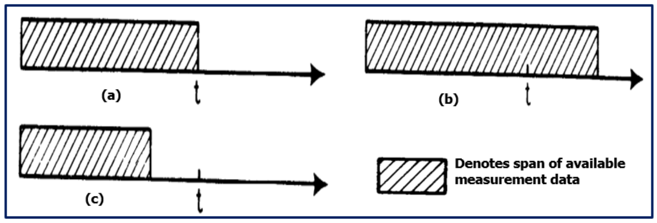

Fig. 1. Pertaining to Q.4.(ii).

Q.5. i). Consider the following adaptive filter model:

Let $\mathbf{x}[\mathbf{n}]$ denote the input vector. Let $\mathbf{h}[\mathbf{n}]$ characterize the unknown system (e.g., channel) and $\widehat{\mathbf{h}}[\mathbf{n}]$ its estimate.

Let $\mathbf{y}[\mathbf{n}] = \mathbf{h}^{\dagger}[\mathbf{n}]\mathbf{x}[\mathbf{n}]$ and $\mathbf{d}[\mathbf{n}] = \mathbf{y}[\mathbf{n}] + \mathbf{I}[\mathbf{n}]$, where $\mathbf{I}[\mathbf{n}]$ denotes the interference $\neq 0$. Further, consider the following definitions:

$$\mathbf{e}[\mathbf{n}] = \mathbf{d}[\mathbf{n}] - \widehat{\mathbf{y}}[\mathbf{n}],\ \delta[\mathbf{n}] = \widehat{\mathbf{h}}[\mathbf{n}] - \mathbf{h}[\mathbf{n}],\ \mathbf{r}[\mathbf{n}] = \widehat{\mathbf{y}}[\mathbf{n}] - \mathbf{y}[\mathbf{n}],\ \boldsymbol{\Delta}_{\mathbf{e}}[\mathbf{n}] = \left|\widehat{\mathbf{h}}[\mathbf{n}] - \mathbf{h}[\mathbf{n}]\right|^{\mathbf{2}}.$$

Consider the following update:

$$\widehat{\mathbf{h}}[\mathbf{n+1}] = \widehat{\mathbf{h}}[\mathbf{n}] + \frac{\mu\, \mathbf{e}[\mathbf{n}]\mathbf{x}[\mathbf{n}]}{\mathbf{x}^{\dagger}[\mathbf{n}]\mathbf{x}[\mathbf{n}]},$$

where μ denotes learning rate.

Derive an expression for optimal learning rate μ^* in terms of σ_r^2, σ_e^2. *Note:* $\mathbf{E}\left[|\mathbf{r}[\mathbf{n}]|^2\right] \triangleq \sigma_r^2$ and $\mathbf{E}\left[|\mathbf{e}[\mathbf{n}]|^2\right] \triangleq \sigma_e^2$. *Hint:* Optimal learning rate minimizes $\mathbf{E}[\boldsymbol{\Delta}_{\mathbf{e}}[\mathbf{n+1}]]$. [4 points]

ii). Consider the following model: $r[n] \sim \mathcal{CN}(0,1)$. Further, $e[n] = 9\sin(n\omega_0 + \Psi)$ is a widesense stationary random process, where $\Psi \sim \mathcal{U}[-\pi \quad \pi]$. Using the formula obtained in (i), compute μ^*. [3 points]

Q.6. Valid or Invalid (5 points) *Note: Write down Valid/Invalid. Justify your response.*

1) Consider the mathematical formulation of compressed sensing (CS): Let $\mathbf{s}$ denote a column vector of weighting coefficients. Let $\mathbf{y}$ denote the vector of measurements. Further, let Φ be a stable measurement matrix, and Ψ is the basis matrix comprising orthonormal basis vectors. The following problem is *concave* in $\mathbf{s}$:

$$\min_{\mathbf{s}} ||\Theta\mathbf{s} - \mathbf{y}||_2 + \lambda||\mathbf{s}||,$$

where $\Theta = \Phi\mathbf{s}$, λ is some real positive constant that weighs importance of sparsity.

2) Recall the optimal filter (Wiener) weights w^* given by $\mathrm{w}^* = \mathcal{R}_x^{-1}\mathrm{r}_{dx}$, where $\mathcal{R}_x$ is the autocorrelation of $x[n]$ and r_{dx} is the crosscorrelation between $d[n]$ and $x[n]$. Consider the quadratic cost $\mathbb{C}_q(\mathrm{w}) = \sigma_d^2 - 2\mathrm{w}^t\mathrm{r}_{dx} + \mathrm{w}^t\mathcal{R}_x\mathrm{w}$. For $\sigma_d^2 = 0.54$, $\mathrm{r}_{dx} = [\frac{1}{\sqrt{8}} \quad -\frac{1}{\sqrt{8}}]^t$ and $\mathrm{w}^* = [\frac{1}{\sqrt{2}} \quad -\frac{1}{\sqrt{2}}]^t$, we have $\mathbb{C}_q(\mathrm{w}^*) = 0.04$.

3) Consider a finite energy pulse $s(t)$. Its magnitude spectrum is given by $|S(f)| = \frac{1}{2\sqrt{\pi}}\exp\left(-4\pi^2 f^2\right)$. The mean square bandwidth $= 0.25$. *Note:* $\int_{-\infty}^{\infty} e^{-\pi f^2}\, df = 1$.

4) Consider a channel impulse response at t_n denoted by $h(t_n) = h_{cn} + jh_{sn}$, is a zero mean, complex Gaussian random process with variance $\sigma^2(t_n)$. The real and imaginary components h_{cn} and h_{sn} are uncorrelated Gaussian random processes each with variance $\sigma^2(t_n)$.

5) Let B be a stable measurement matrix. The standard CS problem is to reconstruct a $\mathbf{s}-$sparse vector $\mathbf{x} \in \mathbb{R}^N$ from the measurements $\mathbf{y} = \mathrm{B}\mathbf{x} \in \mathbb{R}^M$, where $M > N$.

□ END OF QUESTION PAPER □