COMPREHENSIVE EXAMINATION

Instructor-in-charge: B. Sainath, 2210−A, Electrical and Electronics Eng. Dept., BITS Pilani.

Course No./Title : EEE G613/Advanced Digital Signal Processing

DATE: Dec. 18th, 2023. Test duration: 180 Mins. Max. marks: 40.

(*Note*: You may use standard results or formulae by stating them explicitly. Highlight final answers in boxes. Simplify your response to the extent possible. Use notation consistently. Provide key intermediate steps.)

Q.1. [3 + 6 points] *Note:* Provide your answers with accuracy up to four decimal places.

Suppose Y is a real continuous random variable. Consider the following model for the binary Hypothesis detection problem.

$$\mathcal{P}\left(Y;\mathcal{H}_0\right) = \frac{y}{\alpha},\ 0<y<\alpha,\ \alpha\neq 0,$$
$$\mathcal{P}\left(Y;\mathcal{H}_1\right) = \beta y^2,\ 0<y<1,\ \beta\neq 0,\ \alpha<\beta$$

Answer the following:

Part−(a). i) Determine the simplified $N-P$ detector. What is the decision interval (that is, "(lower limit upper limit)") for $\mathcal{H}_1$? *Note:* The decision interval should depend on the threshold γ.

ii). Obtain an inequality on γ, α and β. Extend it further using AM−GM inequality.

Part−(b). i) Let $\widetilde{\gamma} \triangleq \frac{\gamma}{\alpha\beta}$. Determine $\mathcal{P}\left(\text{Error}|\mathcal{H}_0\right)$ in terms of $\widetilde{\gamma}$ and α.

ii). Let $\alpha=1$, $\beta=2$. Determine $\widetilde{\gamma}$ and γ such that $\mathcal{P}\left(\text{Error}|\mathcal{H}_0\right)=1\%$. Determine the decision interval for $\mathcal{H}_1$.

iii). Determine $\mathcal{P}\left(\text{Error}|\mathcal{H}_1\right)$ in terms of $\widetilde{\gamma}$ and β. Using the data given (and obtained) in (ii), compute $\mathcal{P}\left(\text{Error}|\mathcal{H}_1\right)$.

iv). Let $\mathcal{P}\left(\mathcal{H}_0\right)=\mathcal{P}\left(\mathcal{H}_1\right)=0.5$. Determine the average error probability. Assume $\mathcal{P}\left(\text{Error}|\mathcal{H}_0\right)=1\%$ and use the value of $\mathcal{P}\left(\text{Error}|\mathcal{H}_1\right)$ obtained in (iii).

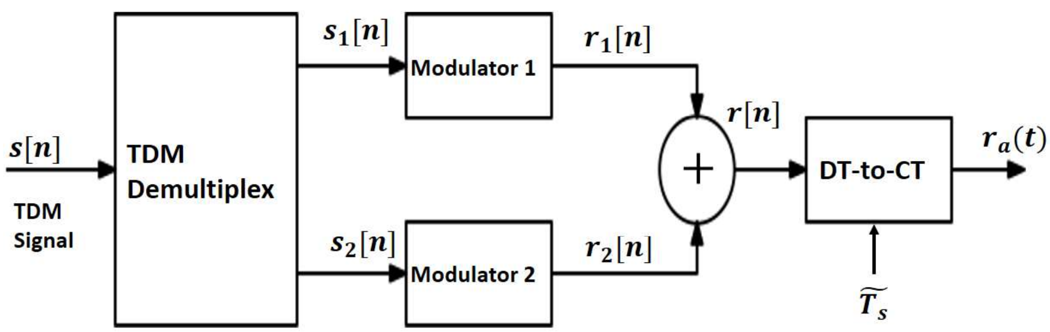

Fig. 1. Pertaining to Q.2.

Q.2. [7 + 2 points] Refer to the TDM−based system block diagram. Consider the following discrete−time sequence $s[n]$, which acts as input to the TDM demultiplex.

$$s[n] = \begin{cases} s_1[0.5n], & n \text{ even},\\ s_2[0.5(n-1)], & n \text{ odd}, \end{cases}$$

Note that $s_1[n]$ and $s_2[n]$ are obtained by sampling $s_{a1}(t)$, and $s_{a2}(t)$, respectively, without aliasing. *Answer the following:*

Part−(a). i). Neatly draw a block diagram of the multirate system that takes $s[n]$ as input and generates $s_1[n]$ and $s_2[n]$. Express $s_1[n]$ and $s_2[n]$ as downsampled (or downsampled and translated) versions of $s[n]$.

ii). Determine whether the multirate system obtained in (i) is linear/non-linear, time-invariant/variant, causal/non-causal, stable/unstable.

iii). Express $S_1[z^2]$ and $S_2[z^2]$ in terms of $S[z]$ and $S[-z]$.

Part−(b). Let $S_{a1}(\Omega)$ is band-limited to $\Omega_{\max}$. Suppose $\Omega_{\max}=10^4\pi$. Compute T_s assuming Nyquist sampling rate. Determine the cutoff frequency ω_{c1} and $\mathcal{L}$ (see the Figure) of a lowpass filter such that after ideal discrete-time (DT) to continuous-time (CT) conversion with sampling period $\frac{T_s}{\mathcal{L}}$. Assume that the Fourier transform of $r_{a1}(t)$ is zero expect $2\pi \times 10^5 \leq |\omega| \leq 2\pi \times 10^5 + 2\Omega_{\max}$. Avoid aliasing. Determine $\widetilde{T}_s = \frac{T_s}{\mathcal{L}}$.

Note/Hints: First determine ω_{c1} in terms of $\mathcal{L}$. Later, write the necessary condition (inequality) that relates ω_{c1}, $\mathcal{L}$ and T_s. Use equality to determine $\mathcal{L}$.

Q.3. [5 + 4 points] Part−(a). Consider the following model:

$$\mathbf{y} = H\psi + \mathbf{w}, \quad \mathbf{y} \in \mathbb{R}^{N\times 1},\ H \in \mathbb{R}^{N\times N},\ \psi \in \mathbb{R}^{N\times 1},\ \mathbf{w} \in \mathbb{R}^{N\times 1},$$

where the Gaussian random vector $\mathbf{w}$ has zero mean Gaussian noise random variables with variance σ_w^2.

Consider the estimator $\widehat{\psi} = \left(H^t H\right)^{-1} H^t \mathbf{y}$, where H^t denotes the transpose of the matrix H. *Assumptions:* Particular realization of H is known. H and $\mathbf{w}$ are statistically independent. *Answer the following:*

i). Determine the mean of the estimator, that is, $\mathbf{E}\left[\widehat{\psi}\right]$. Comment on the estimator.

ii). Determine the covariance of the estimator, that is, $\mathcal{K}_{\widehat{\psi}}$. Comment on the covariance for a particular realization of H, say, $\widehat{H}$.

Part−(b). Suppose we observe a signal subject to multiplicative noise ($\sim$ fading). This random fade may result in a signal that is either 'ON' or 'OFF.' Consider the following model:

$$y[n] = \begin{cases} a_0 + w[n], & n = 0, 1, \ldots, (N_0 - 1), \\ w[n], & n = N_0, N_0 + 1, \ldots, (N-1), \end{cases}$$

with the probability of fade denoted by $p_{\mathcal{F}}$. Assume that we know when a signal has encountered a fade.

Consider the following model: with probability $(1 - p_{\mathcal{F}})$, $\mathbf{h} = \mathbf{1}$ (a column vector of all 1s). Further, with probability $p_{\mathcal{F}}$ $\mathbf{h} = [1\ \ 1\ \ \ldots\ \ 1\ \ 0\ \ 0 \ldots\ \ 0]^t$, where number of 1s is N_0 and remaining are 0s.

Using the results in Part−(a), determine an estimator of a_0 and its variance. Comment on the variance.

Q.4. [4 + 5 points] Consider the following complex-valued function:

$$\mathcal{C}(t) = \sqrt{f_s}\{\text{sinc}(f_s t)\}e^{j2\pi f_c t},\ t \in \mathbb{R},$$

where $\text{sinc}(y) \triangleq \frac{\sin \pi y}{\pi y},\ y \in \mathbb{R}$.

Part−(a). Let $\widehat{C}(f)$ denote the continuous-time Fourier transform (CTFT) of $\mathcal{C}(t)$.

Determine: i). $\widehat{C}(f)$. ii). $\widehat{C}(0)$ and iii). $\int_{-\infty}^{\infty} |C(t)|^2\, dt$. Comment on the usefulness of $\mathcal{C}(t)$ as a mother wavelet.

Note: You may use the properties of CTFT. Explicitly state the properties before applying them.

Part−(b). Consider $\mathcal{C}(t)$. It has the real part denoted by $\mathcal{C}_{\Re}(t)$ and the imaginary part denoted by $\mathcal{C}_{\Im}(t)$. *Answer the following:*

i). Let $\mathcal{C}_{\Re}(t) = \mathcal{C}_{\Re,1}(t) - \mathcal{C}_{\Re,2}(t)$. Determine and simplify $\mathcal{C}_{\Re,1}(t)$ and $\mathcal{C}_{\Re,2}(t)$. Write down the sum and difference frequencies.

ii). Let $\mathcal{C}_{\Im}(t) = \mathcal{C}_{\Im,1}(t) - \mathcal{C}_{\Im,2}(t)$. Determine and simplify $\mathcal{C}_{\Im,1}(t)$ and $\mathcal{C}_{\Im,2}(t)$.

iii). Let $f_s = 1$ Hz and $f_c = 1.5\, f_s$. Neatly sketch $\mathcal{C}_{\Re}(t)$ and $\mathcal{C}_{\Im}(t)$ in the interval $-1 \leq t \leq 1$. *Note:* Take sufficient number of t values to obtain a smooth plot. What is the value of $< \mathcal{C}_{\Re}(t), \mathcal{C}_{\Im}(t) >$?

Q.5. Valid or Invalid (4 points) *Note: Write down Valid/Invalid. Justify your response.*

1) Consider the following equation in the context of compressive sensing.

$$\mathbf{y} = \Theta\mathbf{x}, \quad \mathbf{Y} \in \mathbb{R}^{2\times 1},\ \Theta \in \mathbb{R}^{2\times 3},\ \mathbf{x} \in \mathbb{R}^{3\times 1}.$$

   We get the following information: signal length $= 6$, number of measurements $= 3$.
2) Recall the restricted isometry property (RIP). Consider matrix $\mathcal{B}$ whose RI constant is 0.64. The following holds: $||\mathcal{B}\nu||_2 \geq 0.6$, where ν is a finite norm vector. Assume that ν has unit norm.
3) Consider the two popular algorithms used in adaptive filters, namely, the Least mean squares (LMS) algorithm and recursive least squares (RLS) algorithm. The computational cost of the RLS algorithm is lower than that of the LMS algorithm.
4) Consider the following difference equation:

$$Y[j] = bY[j-1] + \sqrt{1 - b^2}X[j],\ j > -\infty,$$

   where $|b| < 1$, $X[j]$ is a unit−variance, zero mean random process and $Y[j]$ is the output random process.
   Define $\mathcal{R}_X[k] = \mathbf{E}\left[X[j]X[j-k]\right]$. The autocorrelation sequence of $X[j]$ is given by b^k.

□ END OF QUESTION PAPER □