Birla Institute of Technology and Science, Pilani

Advanced Wireless Communications (EEE G614)

Comprehensive Examination, Open Book, First Semester 2023 – 24

Time: 180 minutes — Maximum Marks: 80

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1. Prove the following, given that Λ denotes the LRT, H_0 and H_1 denote the hypothesis, $\mathbb{E}(\cdot)$ denotes the expectation operator, and Var is the variance.
   (a) $\mathbb{E}\left[\Lambda^n | H_1\right] = \mathbb{E}\left[\Lambda^{n+1} | H_0\right]$ **(3)**
   (b) $\mathbb{E}\left[\Lambda | H_0\right] = 1$ **(3)**
   (c) $\mathbb{E}\left[\Lambda | H_1\right] - \mathbb{E}\left[\Lambda | H_0\right] = \text{Var}\left[\Lambda | H_0\right]$ **(3)**

2. If X_1 is Nakagami distributed with the second moment Ω and parameter 4, and X_2 is Rayleigh distributed with second moment $3\,\Omega$, and X_1, X_2 are independent. Find the c.f. of $Y = X_1^2 + 9X_2^2$. **(4)**

3. Packets arrive at a node in a wireless network in two independent simultaneous Poisson streams. The number of packets arriving in a time interval $[t_0, t_0 + T)$ is denoted as $N_1(t_0, t_0 + T)$ for the first stream and by $N_2(t_0, t_0 + T)$ for the second stream. The distribution of $N_i(t_0, t_0 + T)$ is given by

$$\Pr[N_i(t_0, t_0 + T) = k] = \exp(-i\,\lambda\,T)\frac{(i\,\lambda\,T)^k}{k!}, k = 0, 1, 2, ..., \ \lambda > 0, \forall t_0 \geq 0, i = 1, 2.$$

   (a) Calculate the probability that at most one packet arrives at the node in a time interval $\left[\frac{1}{\lambda}, \frac{3}{2\,\lambda}\right)$ **(4)**
   (b) Find the characteristic function of $N_i(0, T)$. **(3)**
   (c) Find the probability $\Pr[N_1(T, 2T), N_2(T, 2T) = k]$ for $k = 0, 1, 2, ...$ **(4)**
   (d) Calculate the rms value of number of packets arriving at the node in a time interval $\left[\frac{3}{2\,\lambda}, \frac{7}{4\,\lambda}\right)$ **(3)**

4. A wireless communication receiver with 2 antennas receives a complex valued transmitted sample $e^{j\Phi}$, where $\Phi \in [-\pi, \pi)$, through in-phase and quadrature channels. In the in-phase and quadrature channels, the 2×1 received signal vectors are $\underline{X}$ and $\underline{Y}$ such that

$$\underline{Z} = \underline{X} + j\underline{Y} = e^{j\Phi}(\underline{G} + j\underline{H}) + \underline{N}_Z\,,$$

where $\underline{G}, \underline{H}$ are random channel gains and $\underline{N}_Z$ is the additive noise, such that

$$\underline{G}, \underline{H} \sim \mathcal{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \Omega\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\right),\ \ \mathbb{E}(\underline{GH}^T) = \Omega\begin{bmatrix}0 & -\rho\\ 1-\rho & 0\end{bmatrix},$$

$$\underline{N}_Z \sim \mathcal{CN}\left(\begin{bmatrix}0\\0\end{bmatrix}, 2\,\sigma^2\underline{I}_2\right),\ \ \Omega, \sigma > 0,\ \ 0 < \rho < 1\,,$$

and $\underline{N}_Z$ is independent of $\underline{G}, \underline{H}$.

   (a) Find $\underline{K}_Z$, the covariance matrix of $\underline{Z}$. **(3)**

(b) For what value of ρ is $\underline{Z}$ circular. **(3)**

(c) Let $\underline{U} = [U_1 \;\; U_2]^T = \underline{X} + \underline{Y}$, $\underline{V} = [V_1 \; V_2]^T = \underline{X} - \underline{Y}$, and $\underline{W} = [W_1 \; W_2]^T = \underline{U} + j\underline{V}$. For the value of ρ in (b), find the c.f.s $\Psi_{W_1,W2}(j\nu_1, j\nu_2)$ and $\Psi_{U_1,V2}(jw_1, jw_2)$. **(4+4)**

5. A zero mean complex circular Gaussian random vector $\underline{Z} = [Z_1 \; Z_2]^T$ has p.d.f.

$$f_{\underline{Z}}(\underline{z}) = \frac{1}{19\,\pi^2} \exp\left\{ -\frac{1}{19}\left(6|z_1|^2 + 4|z_2|^2 + 2\,\mathrm{Re}[(-1+2\,j)z_1^*\,z_2]\right)\right\}, z_1, z_2 \in \mathcal{C}.$$

Let $\underline{X} = \mathrm{Re}(\underline{Z})$ and $\underline{Y} = \mathrm{Im}(\underline{Z})$.

(a) Find the joint characteristic functions $\Psi_{X_1,Y_2}(jw_1, jw_2)$ and $\Psi_{Y_1,X_2}(jw_1, jw_2)$ **(3+3)**

(b) Calculate $\mathbb{E}[X_1\,X_2\,Y_1\,Y_2]$. **(3)**

(c) Find the joint p.d.f. $f_{X_1,X_2}(x_1, x_2)$ and $f_{X_1,Y_1}(x_1, y_1)$ **(3+3)**

(d) Let $R = \sqrt{X_1^2 + Y_1^2}$. Find the characteristic function and c.d.f. of R^2. **(2+2)**

6. A WSS complex-valued circular Gaussian random process $Z(t)$ is expressed as

$$Z(t) = X(t) + jY(t)\,,$$

where $j = \sqrt{-1}$ and $X(t), Y(t)$ are real-valued jointly WSS Gaussian random processes. The random process $Z(t)$ has mean $\mu_Z = \mu_X + j\mu_Y$, with $\mu_X = \mathrm{Re}(\mu_Z) > 0$, $\mu_Y = \mathrm{Im}(\mu_Z) > 0$, autocorrelation function

$$R_Z(t_1, t_2) = \mathbb{E}[Z(t_1)Z^*(t_2)] = \left\{(256)^{-|t_1-t_2|}\cos\left(4\,\pi(t_1 - t_2)\right)\right\} + |\mu_Z|^2\,,$$

and pseudo-autocorrelation function

$$\tilde{R_Z}(t_1, t_2) = \mathbb{E}[Z(t_1)Z(t_2)] = 2 + j(2\sqrt{3})\,.$$

Find the following:

(a) the mean μ_Z, the cross-covariance function

$$K_{XY}(t_1, t_2) = \mathbb{E}[(X(t_1) - \mu_X)(Y(t_2) - \mu_Y)]\,,$$

the autocovariance function

$$K_X(t_1, t_2) = \mathbb{E}[(X(t_1) - \mu_X)(X(t_2) - \mu_X)]\,,$$

and the autocorrelation function $R_Y(t_1, t_2) = \mathbb{E}[Y(t_1)Y(t_2)]$. **(3+3+2+2)**

(b) Let $V = V_1 + jV_2 = Z(0) + 2Z(\frac{1}{4}) + 3Z(\frac{1}{2})$, where $V_1 = \mathrm{Re}(V)$ and $V_2 = \mathrm{Im}(V)$. Find the joint p.d.f. $f_{V_1,V_2}(v_1, v_2)$ and the c.f. of $V_1^2 - 3V_2^2$. **(5+5)**