# COMPREHENSIVE EXAMINATION

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Course No.: ADC/EEE-G622

DATE: May 1st, 2018 (AN) MAX. TIME: 3 hrs.

**Note:** You should use the same notation or the symbols given in each question. Do not skip intermediate steps. Simplify your answers to the maximum extent possible. Highlight your final answer in rectangular box.

**Q. 1. [On Ratio of Fading Amplitudes] [9 points]**

Consider two statistically independent, frequency-flat Rayleigh fading channels. Their fading coefficients are modeled as $h_1 \sim \mathcal{CN}(0,1)$ and $h_2 \sim \mathcal{CN}(0,1)$, that is, circularly symmetric complex Gaussian random variables (CSCG). Answer the following:

i). Let $X = |h_1|$ and $Y = |h_2|$. Write down the probability density function (PDF) of $X$ and $Y$. [1 point]

ii). Determine the cumulative distribution function (CDF) and the PDF of the random variable $Z = \frac{X}{Y}$, and sketch them. [3 + 1 + 1 points]

iii). Compute the probability of the following event: $\mathcal{P}\left(\frac{|h_1|}{|h_2|} > 0 \text{ dB}\right)$. [2 points]

iv). Compute the probability of the following event: $\mathcal{P}\left(\frac{|h_1|^2}{|h_2|^2} < 0 \text{ dB}\right)$. [1 point]

**Q. 2. [On PAPR Problem in OFDM] [5 points]**

Let $d[n]$ denote time-domain samples which can be computed from the following inverse DFT (IDFT) formula.

$$d[n] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} d[k] \exp\left(j2\pi k \frac{n}{N}\right), 0 \leq n \leq N-1,$$

where N denotes the number of subcarriers, $j = \sqrt{-1}$. Suppose that $d[n] \sim \mathcal{CN}(0,1), 0 \leq n \leq N-1$.

i). Let $\mu$ denote the PAPR threshold. Derive an expression for $\mathcal{P}(\text{PAPR} \leq \mu)$, denoted by $p$. Compute the probability if the length of $d[n]$ is 4 and $\mu = 1$ dB. [2 + 1 points]

ii). If $N_2 > N_1$, plot $\mathcal{P}(\text{PAPR} \leq \mu)$ as a function of $\mu$ in dB. Show the illustration for $N_1 = 16$ and $N_2 = 64$ and $\mu$ ranging from 0 to 10 dB. Comment on the impact of large N on the probability $p$. [2 points]

**Q. 3. [On Power Density Spectrum: Clarke's Model] [9 points]** Consider the Clarke's model shown in the Figure 1. Assume N uniform isotropic scatterers. Doppler of n^th path is given by $f_{D_n} = \frac{\text{v}}{\lambda}\cos(n\Delta\theta)$, where v denotes the speed of receiving antenna, $\lambda$ is the carrier wavelength, and $\Delta\theta = \frac{2\pi}{N}$.

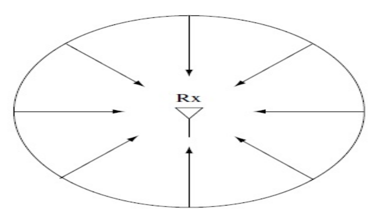


Fig. 1: Figure pertaining to Q. 3.

The impulse response $h(t)$ is given by

$$h(t) = \sqrt{\frac{P_r}{N}} \sum_{n=1}^{N} \exp\left(j\phi_n(t)\right),$$

where $\phi_n(t) = 2\pi f_{D_n} t$, $\sqrt{\frac{P_r}{N}}$ denotes the amplitude from each multipath component. Answer the following:

i). Determine the autocorrelation function (ACF) $\mathcal{R}_{hh}(\tau)$. Assume wide-sense stationary (WSS) random process. What is the significance of $R_{hh}(\tau)$?. Recall that the ACF is defined as $R_{hh}(\tau) = \mathbf{E}\left[h^\star(t)h(t-\tau)\right]$, where $\star$ denotes conjugate. The following definition is useful: The zeroth order Bessel function is defined as

$$J_0(y) = \frac{1}{\pi}\int_0^\pi \exp\left(-jy\cos\theta\right)\, d\theta.$$

ii). Determine power density spectrum $\mathcal{S}_{hh}(f)$ using the fact that ACF and power spectral density (PSD) forms a Fourier transform pair. Give at least two useful remarks on the result obtained. [4 + 1 + 4 points]

**Q. 4. [On Wireless Channel Fading][7 points]**

i). Consider a wireless fading scenario. How do you model when the direct component of the transmitted signal is shadowed, and simultaneously exist with scattered components? Justify. Give a brief description of the model qualitatively and quantitatively. [2.5 points]

ii). Consider a Gaussian random variable (RV) $Y$ with mean $-2.2724$ and standard deviation $1.6469$. Compute mean and variance of $Z = \exp(Y)$.

Consider a 100% shadowed transmitted signal path characterized by the distribution of $Z$ in (ii). Compute the percentage of time the received signal power will fade below 9 dB. [2 + 2.5 points]

**Q. 5. [On Spectral characterization of narrowband fading channel][6 points]**

*Notation:* rect $\left(\frac{f}{B}\right)$ denotes rectangular spectrum which has constant amplitude of 1 in the interval $[-\frac{B}{2}, \frac{B}{2}]$.

Consider a narrowband wireless channel with carrier frequency 900 MHz. The complex baseband channel fading coefficient $h(t) = g_I(t) + jg_Q(t)$, where $g_I(t)$, and $g_Q(t)$ denote the in-phase, and quadrature random processes, respectively. The power spectral densities $S_{g_I g_I}(f)$ and $S_{g_I g_Q}(f)$ are given by

$$S_{g_I g_I}(f) = \begin{cases} \text{rect}\left(0.005f\right), & |f| < 100Hz, \\ 0, & \text{elsewhere}, \end{cases}$$

and, $S_{g_I g_Q}(f) = 0$, respectively.

*Answer the following:*

i). Compute the speed of the mobile station in kmph. Assume angle of arrival $\theta = 0°$. [1 point]

ii). Determine the cross-correlation function (CCF) $R_{IQ}(\tau)$ of the in-phase and quadrature components of the faded envelope? Comment on your result in one or two sentences. [1.5 points]

iii). Recall the usefulness of space diversity. It can be achieved by employing multiple antennas at the base station and/or at the mobile station (MS). If antenna diversity is used at the MS, how much should be the distance between antennas elements so that the corresponding fading envelopes are uncorrelated? [2.5 points]

iv). Determine the expression of the autocorrelation function $R_{II}(\tau)$. [1 point]

**Q. 6. [TRUE/FALSE] [4 points]**

Each question carries 1 point. Indicate 'TRUE' for true statement and 'FALSE' for false statement. Justify your answer in just ONE or TWO sentences.

i). The key benefit of cooperative diversity arises from the broadcast nature of the wireless channel, along with its ability to achieve diversity through statistically non-independent fading channels.

ii). Suppose that the cognitive radio (CR) operates in interweave paradigm. The stringent need is that the secondary (or unlicensed) users should not interfere with the communication between the active primary users.

iii). Consider an RF energy harvesting sensor network. In it, energy constrained nodes can scavenge energy and process the information simultaneously

iv). While signals at lower frequency bands (for example, GSM signals) can propagate for several kilometers and more easily penetrate buildings, millimeter wave (mmWave) signals can only travel a few kilometers, or less.

ALL THE BEST!

# I. SOLUTIONS

**Q. 1.** i). The random variable (RV) $X = |h_1|$ is Rayleigh distributed with PDF $p_X(x) = 2x\exp\left(-x^2\right), x \geq 0$. Similarly, the random variable (RV) $Y = |h_2|$ is Rayleigh distributed with PDF $p_Y(y) = 2y\exp\left(-y^2\right), y \geq 0$.

ii). The CDF is given by

$$F_Z(z) = \frac{z^2}{z^2+1}, z \geq 0.$$

The derivative of the CDF, that is, the PDF is given by

$$p_Z(z) = \frac{2z}{\left(z^2+1\right)^2}, z \geq 0.$$

The plots of the CDF and the PDF are shown in the Figure 1.

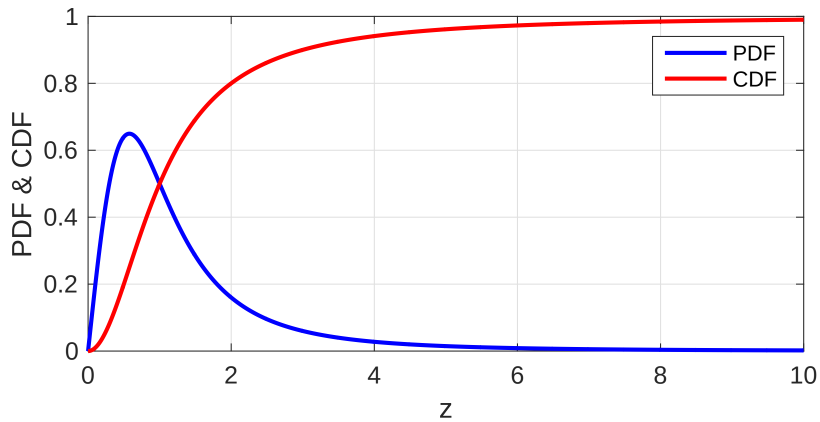


Fig. 2: The CDF and the PDF of the random variable $Z = \frac{X}{Y}$.

iii). Probability $= \frac{1}{2}$.

iv). Probability $= \frac{1}{2}$.

**Q. 2.** i). Expression can be derived as follows.

$$\begin{aligned}\mathcal{P}(\text{PAPR} \leq \mu) &= \mathcal{P}(|d[0]|^2 \leq \mu, \ldots, |d[n-1]|^2 \leq \mu),\\ &= \left\{\mathcal{P}\left(|d[0]|^2 \leq \mu\right)\right\}^N,\\ &= \left(1-e^{-\mu}\right)^N \triangleq p.\end{aligned}$$

The final expression is due to the fact that $|d[0]|^2 \sim \exp(1)$.

Therefore, we observe the following: i). As the PAPR threshold increases, the probability $p$ increases. ii). For a fixed threshold, as $N$ increases, the probability $p$ decreases.

The desired probability $\mathcal{P}(\text{PAPR} \leq \mu) = \left(1-e^{-\mu}\right)^4 = 0.2629$.

ii). The plots for scenario in which $N_2 > N_1$ are shown below.

From the Figure 3, we observe that probability $p \to 0$ as $N \to \infty$.

**Q. 3.** i). ACF is given by $\mathcal{R}_{hh}(\tau) = P_r J_0\left(2\pi f_D \tau\right)$.

ii). PSD is given by $\mathcal{S}_{hh}(f) = \frac{P_r}{\pi f_D}\frac{1}{\sqrt{1-\left(\frac{f}{f_D}\right)^2}}, |f| \leq f_D$.

*Remarks:* i). PSD $\to \infty$ as $f \to f_D$ or $f \to -fD$. ii). At $f = 0$, PSD $= \frac{P_r}{\pi f_D}$.

**Q. 4.** i). The model: Rican plus log-normal distribution. Fading factor $K$ of Rican model for small scale fading characterizes the strength of the direct component relative to the scatters components. On the other hand, large

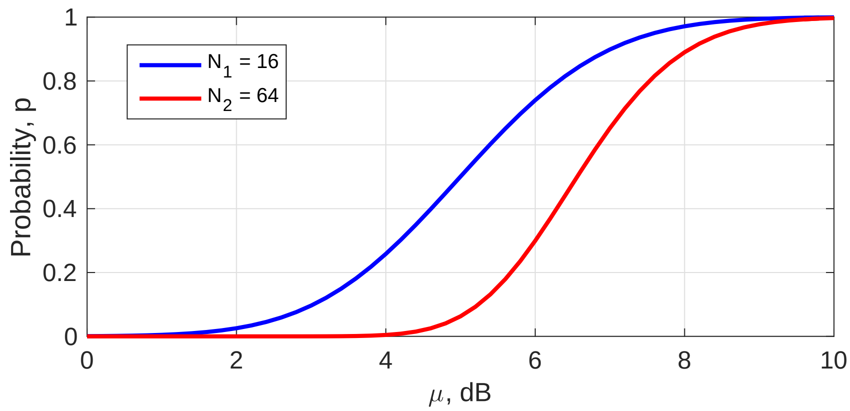


Fig. 3: Probability $p$ as a function of PAPR threshold $\mu$ in dB.

scale fading effects are accounted by log-normal random variable. Since Rican and log-normal random variables are statistically independent, their joint pdf characterizes the overall fading.

ii). *Mean:* $\mu_Z = \exp\left(\mu_Y + \frac{\sigma_Y{}^2}{2}\right) = 0.4$.

*Variance:* $\sigma_Z{}^2 = \exp\left(2\mu_Y + 2\sigma_Y{}^2\right) - \mu_Z{}^2 = 2.25$ and the standard deviation is $1.5$.

iii). The desired probability is given by

$$\mathcal{P}(Z < 10^{0.9}) = \frac{1}{\sqrt{2\pi}1.5}\int_0^{7.9433} \frac{1}{z}\exp\left(-\frac{(\ln z - 0.4)^2}{4.5}\right) dz.$$

Use substitution method to simplify the definite integral. This approach yields probability $1 - Q\left(\frac{\ln(7.9433)-0.4}{1.5}\right)$. Simplifying further yields $0.8676$.

**Q. 5.** i). Let $f_m$ denote the maximum Doppler shift. $v = \frac{f_m c}{f_c} \approx 33.4$ m/s or $120.4$ kmph.

ii). The CCF is nothing but the inverse FT of the cross spectral density. Therefore, $R_{IQ}(\tau) = 0$. This is because, by invoking the central limit theorem, we see that the in-phase and quadrature components are Jointly Gaussian. Therefore, they are also individually Gaussian and uncorrelated.

iii). Show that $R_{II}(\tau) = 2f_m\text{sinc}(2f_m\tau)$. Since $f_m\tau = \frac{d}{\lambda_c}$, we get $R_{II}(\tau) = 2f_m\text{sinc}(2\frac{d}{\lambda_c})$ = 0 when $\frac{2\pi d}{\lambda_c} = n\pi$. In other words, $\frac{d}{\lambda_c} = \frac{n}{2}$, that is, multiple of a half wavelength.

iv). $R_{II}(\tau) = 200\,\text{sinc}(200\tau)$. Recall the definition: $\text{sinc}(t) = \frac{\sin\pi t}{\pi t}$.

**Q. 6.** i). *False*. To achieve diversity, fading channels should be statistically independent.

ii). *True.* The secondary users should be able to detect (sense), with very high probability, the primary user transmissions in the cognitive radio network (CRN).

iii). *True.* Because, RF signals can carry energy and information at the same.

iv). *True.* Because, mmWaves suffer from high transmission losses in the air and solid objects. However, their propagation characteristics can be very advantageous in some applications such as wireless personal area networks (WPANs).