MID-SEMESTER TEST

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Course: Advanced Digital Communication (EEE G622)

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(*Note*: You may use standard results or formulas. However, state them clearly and precisely. Highlight your final answers in rectangular boxes.)

Q. 1. Consider the following random process:

$$Y(t) = \mathcal{R} \sin(2\pi f_c t + \Theta) + X(t),$$

where i). f_c is some positive constant, ii). $\Theta \sim \mathbb{U}[0, 2\pi)$, iii). $\mathcal{R}$ denotes Rayleigh distributed random variable. Its probability density function $p_{\mathcal{R}}(r)$ is given by

$$p_{\mathcal{R}}(r) = \begin{cases} \frac{2r}{b} \exp\left(-\frac{r^2}{b}\right), & r \geq 0, \\ 0, & r < 0. \end{cases}$$

iv). $X(t)$ is a wide sense stationary (WSS) random process with mean μ_X, and autocorrelation function $\mathfrak{R}_X(\tau)$, respectively. Furthermore, $X(t)$ and $\mathcal{R}$ are statistically independent.

Answer the following:

a). Let $t_1 - t_2 \triangleq \tau$. Determine the autocorrelation function $\mathfrak{R}_Y(t_1, t_2) = \mathbf{E}[Y_{t_1} Y_{t_2}]$.

b). If $X(t)$ is the AWGN process, determine $\mathfrak{R}_Y(\tau)$ and its power spectral density (PSD). [6 points + 3 points]

Q. 2. Consider the constellation shown in the Figure 1. *Answer the following:*

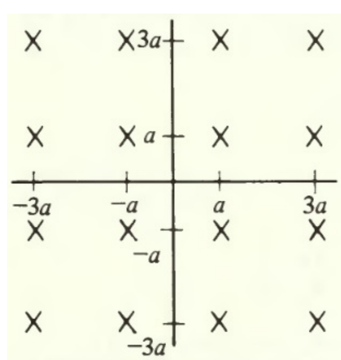

Fig. 1: Constellation diagram for Q. 2.

a). Determine ‘a’ such that the average energy of each symbol, denoted by E_s, is equal to 1. [1 point]

b). Determine the optimum decision regions. Label the regions $\Lambda_1, \ldots, \Lambda_{16}$. *Note:* Label from left to right, starting from the left topmost signal point. [1 point]

c). Suppose that the symbols shown in the constellation are transmitted over the AWGN channel. Derive the average probability of correct decision $p_{\mathcal{C}}$ and the average symbol error probability (ASEP) $p_{\mathcal{E}}$ in terms of average energy to noise PSD ratio. Assume that all symbols are equiprobable and the noise components of in-phase and quadrature axes are independent with variance $\frac{N_0}{2}$. (*Hint:* Identify identical decision regions. Group them and then determine the probability of correct decision group-wise.) [6 points]

d). Suppose that the noise PSD (two-sided) is -10 dBm and average symbol energy is 0.01 joule. Compute $p_{\mathcal{E}}$. [1 point]

Q. 3. Consider the following random process.

$$Z(t,\omega) = X(\omega)\cos\left(2\pi f_c t\right) + Y(\omega)\sin\left(2\pi f_c t\right), \omega \in \mathcal{R}, t \in \mathcal{R},$$

where $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$. Note that X and Y are statistically independent.

a). Determine the mean and the variance of $Z(t)$. Comment on the nature of the random process $Z(t)$ based on the result. [4 points]

b). Suppose that we sample $Z(t)$ at two different instants t_1, t_2. We obtain the random variables Z_{t_1} and Z_{t_2}. Determine autocovariance function $\mathfrak{K}_Z(t_1, t_2)$. Based on the result, comment on the nature of the random process $Z(t)$. [4 points]

c). Express autocovaraince as a function of t and τ, where $t_1 = t$ and $t_2 = t - \tau$. [1 point]

Q. 4. (True/False) Each question carries 1 point. Indicate 'T/F' ('T' for True statement; 'F' for false statement.). Give proper justification in one or two sentences. [3 points]

a). The minimum achievable probability of error is not influenced by the introduction of a reversible operation at the output of a channel.

b). Natural logarithm of the Bhattacharyya bound gives the Bhattacharyya distance, which measures the similarity of two discrete or continuous probability distributions.

c). Consider the following periodic function $Y(t)$.

$$Y_p(t) = \sum_{j=-\infty}^{\infty} r(t - jT + k\Phi),$$

where k is a constant, Φ is a uniform random variable, and $r(t)$ is the ramp function defined in the interval $[0, T]$.

$Y_p(t)$ is a stationary random process when $k = 0$.

Answers

Q. 1. a). The autocorrelation function of $Y(t)$ is given by

$$\mathfrak{R}_Y(t_1,t_2) = \mathbf{E}\left[\frac{\mathcal{R}^2}{2}\left(\cos\left(2\pi f_c(t_1-t_2)\right) - \cos\left(2\pi f_c(t_1+t_2)+2\Theta\right)\right)\right]$$
$$+\,\mathbf{E}\left[R\right]\mathbf{E}\left[X(t_1)\right]\mathbf{E}\left[\sin\left(2\pi f_c t_1+\Theta\right)\right] + \mathbf{E}\left[R\right]\mathbf{E}\left[X(t_2)\right]\mathbf{E}\left[\sin\left(2\pi f_c t_2+\Theta\right)\right] + \mathbf{E}\left[X_{t_1}X_{t_2}\right].$$

To simplify further, we use the following.

$$\mathbf{E}\left[X_{t_1}X_{t_2}\right] = \mathfrak{R}_X(t_1,t_2),$$
$$\mathbf{E}\left[\sin\left(2\pi f_c t_1+\Theta\right)\right] = 0,$$
$$\mathbf{E}\left[\sin\left(2\pi f_c t_2+\Theta\right)\right] = 0,$$
$$\mathbf{E}\left[\cos\left(2\pi f_c(t_1+t_2)+2\Theta\right)\right] = 0.$$

Substituting the above, we get

$$\mathfrak{R}_Y(t_1,t_2) = \frac{1}{2}\mathbf{E}\left[\mathcal{R}^2\right]\cos\left(2\pi f_c(t_1-t_2)\right) + \mathfrak{R}_X(t_1,t_2).$$

Recall that $X(t)$ is WSS. Therefore, we have $\mathfrak{R}_X(t_1,t_2) = \mathfrak{R}_X(\tau)$. Furthermore, since $\mathcal{R}$ is Rayleigh distributed, we can show that $\mathbf{E}\left[\mathcal{R}^2\right] = b$. Simplifying further, we get

$$\mathfrak{R}_Y(\tau) = \frac{b}{2}\cos\left(2\pi f_c\tau\right) + \mathfrak{R}_X(\tau).$$

b). Recall that the autocorrelation function of AWGN is $\frac{N_0}{2}\delta(\tau)$. Therefore, we get

$$\mathfrak{R}_Y(\tau) = \frac{b}{2}\cos\left(2\pi f_c\tau\right) + \frac{N_0}{2}\delta(\tau).$$

Since the PSD is the Fourier transform of the ACF, we get

$$\mathfrak{S}_Y(f) = \frac{b}{4}\left(\delta\left(f-f_c\right) + \delta\left(f+f_c\right)\right) + \frac{N_0}{2}.$$

Q. 2. a). Average energy per symbol $E_s = 10a^2 = 1$. Therefore, $a = \frac{1}{\sqrt{10}}$.

b). The optimum decision regions are shown in the Figure 2.

c). Let $n \sim \mathcal{N}(0, \frac{N_0}{2})$. Consider $\mathcal{P}(n > a) = Q\left(\sqrt{\frac{2a^2}{N_0}}\right) = Q\left(\sqrt{\frac{1}{5N_0}}\right) \triangleq \mathcal{C}$.

Decision regions can be divided into the following three groups.

Group I: $\{\Lambda_1, \Lambda_4, \Lambda_{13}, \Lambda_{16}\}$.

Group II: $\{\Lambda_2, \Lambda_3, \Lambda_5, \Lambda_8, \Lambda_9, \Lambda_{12}, \Lambda_{14}, \Lambda_{15}\}$.

Group III: $\{\Lambda_6, \Lambda_7, \Lambda_{10}, \Lambda_{11}\}$.

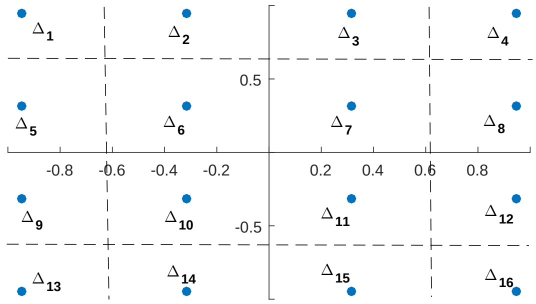

Fig. 2: Optimum decision regions of the given constellation.

Probability of correct decision: Recall that $y_1 = x_1 + n_1$, $y_2 = x_2 + n_2$. We have $p_{\mathcal{C}}(\Lambda_4) = \mathcal{P}(y_1 > 2a, y_2 > 2a | x_1 = 3a, x_2 = 3a) = \mathcal{P}(n_1 > -a, n_2 > -a) = \mathcal{P}(n_1 < a, n_2 < a) = \left(1 - Q\left(\sqrt{\frac{2a^2}{N_0}}\right)\right)^2 = (1-\mathcal{C})^2 = p_{\mathcal{C}}(\Lambda_1) = p_{\mathcal{C}}(\Lambda_{13}) = p_{\mathcal{C}}(\Lambda_{16})$.

Similarly, we can show the following: $p_{\mathcal{C}}(\text{Group II}) = (1-\mathcal{C})(1-2\mathcal{C})$ and $p_{\mathcal{C}}(\text{Group III}) = (1-2\mathcal{C})^2$. Therefore, the average probability of correct decision is given by

$$p_{\mathcal{C}} = \frac{1}{16}\left(4(1-\mathcal{C})^2 + 8(1-\mathcal{C})(1-2\mathcal{C}) + 4(1-2\mathcal{C})^2\right) = \left(1 - \frac{3\mathcal{C}}{2}\right)^2.$$

Finally, the average symbol error probability (ASEP) is given by

$$p_{\mathcal{E}} = 1 - \left(1 - \frac{3\mathcal{C}}{2}\right)^2 = 3\mathcal{C} - \frac{9\mathcal{C}^2}{4}.$$

To express $p_{\mathcal{E}}$ In terms of average symbol energy to noise PSD ratio, $\mathcal{C} = Q\left(\sqrt{\frac{2a^2}{N_0}}\right) = Q\left(\sqrt{\frac{E_s}{10\sigma^2}}\right)$, where $\sigma^2 = \frac{N_0}{2}$.

d). Given that $E_s = 10a^2 = 0.01$ joule and $\frac{N_0}{2} = -10$ dBm $= -40$ dB. Substituting these values, we get $\mathcal{C} = Q(\sqrt{10})$. Therefore, $p_{\mathcal{E}} = 0.0023$.

Q. 3. a). Given that both X and Y are Gaussian random variables. Therefore, the random process $Z(t)$ has mean given by

$$\mu_Z(t) = \mu_X \cos(2\pi f_c t) + \mu_Y \sin(2\pi f_c t),$$

and variance is given by

$$\sigma_Z{}^2(t) = \sigma_X{}^2 \cos^2(2\pi f_c t) + \sigma_Y{}^2 \sin^2(2\pi f_c t).$$

Since the mean and the variance of $Z(t)$ are periodic in the t, the random process is first-order cyclostationary.

b). Autocovariance of $Z(t)$ is given by

$$\begin{aligned}\mathfrak{K}_Z(t_1,t_2) &= \mathbf{E}\left[Z_{t_1}Z_{t_2}\right] - \mu_{Z_{t_1}}\mu_{Z_{t_2}},\\ &= {\sigma_X}^2\cos\left(2\pi f_c t_1\right)\cos\left(2\pi f_c t_2\right) + {\sigma_Y}^2\sin\left(2\pi f_c t_1\right)\sin\left(2\pi f_c t_2\right),\\ &= \frac{{\sigma_X}^2+{\sigma_Y}^2}{2}\cos\left(2\pi f_c(t_1-t_2)\right) + \frac{{\sigma_X}^2-{\sigma_Y}^2}{2}\cos\left(2\pi f_c(t_1+t_2)\right).\end{aligned}$$

c). For $t_1 = t$ and $t_2 = t_1 - \tau$, Autocovariance of $Z(t)$ is given by

$$\mathfrak{K}_Z(t,\tau) = \frac{{\sigma_X}^2+{\sigma_Y}^2}{2}\cos\left(2\pi f_c\tau\right) + \frac{{\sigma_X}^2-{\sigma_Y}^2}{2}\cos\left(4\pi f_c t - 2\pi f_c\tau\right).$$

Q. 4. a). TRUE. This is the theorem of reversibility. Reversible operations such as translation, rotation will not affect the error performance.

b). FALSE. Negative of natural logarithm of the Bhattacharyya bound gives the Bhattacharyya distance D_B. If $p_1(y)$ and $p_2(y)$ denote the probability density functions, $D_B \triangleq -\ln\rho$, where $\rho = \int\sqrt{p_1(y)p_2(y)}\,dy$.

c). FALSE. If $k=0$, $Y_p(t)$ becomes deterministic and periodic signal. Therefore, it is not a stationary random process.