Birla Institute of Technology and Science, Pilani

Applied Estimation Theory (EEE G641)

Comprehensive Examination, First Semester 2022 – 23

Time: 180 minutes | Maximum Marks: 80

1. Given, when $A > 0$,

$$\mathcal{H}_0 : p(x) = \frac{1}{2}\exp(-|x|)$$
$$\mathcal{H}_1 : p(x) = \frac{1}{2}\exp(-|x - A|)$$

   (a) Find the ML test to distinguish between $\mathcal{H}_0$ and $\mathcal{H}_1$. What is P_D and P_{FA}. **(6)**

   (b) Find the NP test (to distinguish between $\mathcal{H}_0$ and $\mathcal{H}_1$) that achieves $P_{FA} = 0.1$. What is P_D? Does your test depend on the value of A? Explain. **(6)**

2. The random variable x is $\mathcal{N}(m, \sigma)$. It is passed through one of the two non-linear transformations, find the LRT. **(6)**

$$\mathcal{H}_1 : y = \exp(x)$$
$$\mathcal{H}_0 : y = x^2$$

3. We want to transmit two parameters A_1 and A_2, to achieve a secure transmission, the signals to be transmitted over separate channels are constructed as

$$s_1 = x_{11}A_1 + x_{12}A_2\,,$$
$$s_2 = x_{21}A_1 + x_{22}A_2\,,$$

where $x_{ij}, i, j = 1, 2$, are known. The received variables are

$$r_1 = s_1 + n_1\,,$$
$$r_2 = s_2 + n_2\,.$$

The additive noises are i.i.d., zero-mean Gaussian random variables, $\mathcal{N}(0, \sigma_n)$. The parameters A_1 and A_2 are nonrandom.

   (a) Check whether the ML estimates $\hat{a}_1$ and $\hat{a}_2$ are unbiased. **(5)**

   (b) Compute the variance of the ML estimates $\hat{a}_1$ and $\hat{a}_2$. **(5)**

   (c) Check whether the ML estimates are efficient. **(5)**

4. Consider the general Gaussian binary hypothesis testing problem $\mathcal{H}_1 : \mathbf{Y} = \mathbf{X} + \mathbf{N}$ and $\mathcal{H}_0 : \mathbf{Y} = \mathbf{N}$ with $m_0 = 0$, covariance matrix $\mathbf{K} = \begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 1 \end{bmatrix}$ and $\mathbf{m}_1 = \begin{bmatrix} m_{11} \\ m_{12} \\ m_{13} \end{bmatrix}$. Obtain the sufficient statistic in the new coordinate system where the components of the observation vector are independent. **(8)**

5. Consider a binary communication system transmitting equiprobable information symbols with ASK, the received signals under the two hypothesis are

$$\mathcal{H}_1 : y(t) = A\, s(t) + w(t),\ 0 \leq t \leq T$$
$$\mathcal{H}_0 : y(t) = w(t),\ 0 \leq t \leq T$$

The attenuation A produced by the communication channel is a Gaussian random variable with mean zero and variance σ_a^2. The signal $s(t)$ is deterministic with energy E and $w(t)$ is an AWGN with mean zero and psd $N_0/2$. Determine optimum receiver, assuming minimum probability of error criteria. **(10)**

6. You are looking for underground coal deposits by transmitting sound waves and measuring the return response. When there is no deposit, the return signal x has a Gaussian distribution with a mean of 8 and variance of 1. When there is a deposit, the distribution of the return signal x is a again Gaussian, but the mean is 10 with a variance of 2. If you decide that there is a coal deposit, you will excavate the area at a cost of Rs. 10 Lakhs to extract the coal. You know that the value of each coal deposit is Rs. 110 Lakhs. You have hired a consultant, and this person tells you that you should expect 20% of the areas that you test in this region to have coal deposits.

   (a) Construct a decision rule that tells you (on the basis of the measurement x) whether or not to initiate an excavation. Your rule should maximize your expected net gain. **(8)**

   (b) Now assume that the expected value of the coal deposits is Rs. 20 Lakhs. What is the decision rule in this case? Explain. **(5)**

7. Design a minimum P_e detector to decide among the hypotheses whose PDFs are

$$p\,(x(0)\,|\,\mathcal{H}_0) = \frac{1}{2}\exp(-|x(0)+1|)$$
$$p\,(x(0)\,|\,\mathcal{H}_1) = \frac{1}{2}\exp(-|x(0)|)$$
$$p\,(x(0)\,|\,\mathcal{H}_2) = \frac{1}{2}\exp(-|x(0)-1|)$$

assuming equal a priori probabilities. What is the minimum value of P_e. **(8)**

8. You observe i.i.d. data $x(n)$ for $n = 0, 1, ..., N-1$ which consist only of noise, but you want to detect whether the noise is non-Gaussian. The null hypothesis $\mathcal{H}_0$ is Gaussian noise, and you model the alternate hypothesis $\mathcal{H}_1$ as being Laplacian noise. For each $x(n)$, then, we have

$$\mathcal{H}_0 : p(x(n)) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{x^2(n)}{2\sigma^2}\right)$$
$$\mathcal{H}_1 : p(x(n)) = \frac{1}{\sqrt{2\sigma^2}}\exp\left(-\sqrt{\frac{2}{\sigma^2}}\,|x(n)|\right)$$

for $n = 0, 1, ..., N-1$. The variance of the noise in both cases is σ^2, but this parameter is unknown. Find the GLRT statistic $L_G(x)$. **(8)**

End of Paper