Birla Institute of Technology & Science, Pilani
First Semester 2023-2024, MATH F111 (Mathematics-I)
Comprehensive Examination (PART-A Closed Book)

Max. Marks: 135 **December 18, 2023** **Time: 180 Min.**

---

1. The question paper consists of two parts: **Part-A (Closed Book)** and **Part-B (Open Book). Part-A** is of 66 Marks & 90 Minutes and **Part-B** is of 69 Marks & 90 Minutes. Attempt questions of **Part-A** and **Part-B** on two separate answer sheets.
2. The Question 1 of the Closed Book must be answered on the first page of the answer-sheet otherwise it will not be evaluated.
3. Notations/symbols have their usual meaning.
4. Start answering each question on a new page and answer the sub-parts of each question in continuation. Write **END** at the end of the last attempted question in each answer sheet.

Max. Marks: 66 **PART-A(Closed Book)** **Time: 90 Min.**

---

1. Give the most appropriate answer to the following questions. **Answer this question of the first page of the answer-sheet otherwise it will not be evaluated.** **[2×7=14]**

   (a) A point within a triangle such that the sum of the square of its distances from three vertices is minimum is:
   (A) Circumcenter (the point which is equidistant from the three vertices)
   (B) Centroid (the point whose coordinates are the averages of the corresponding coordinates of the vertices of the triangle)
   (C) Orthocenter (the point where the perpendicular drawn from the vertices to the opposite sides of the triangle intersect each other)
   (D) None of the above.

   (b) The value of $\int_C (\cos(x)\,dx + \sin(y)\,dy)$, where C is a curve given by $-t\,\mathbf{i} + (\cos^2(t) - \cos^2(1))\,\mathbf{j}$; $-1 \leq t \leq 1$ is:
   (A) $\sin(1)$ (B) $\sin^2(2) - \cos^2(1)$ (C) $\sin^2(2) + \cos^2(1)$ (D) None of the above.

   (c) The value of the integral $\int_0^1 \int_x^1 e^{y^2}\,dy\,dx$ is:
   (A) $\frac{(e-1)}{2}$ (B) $\frac{(e+1)}{2}$ (C) $\frac{(1-e)}{2}$ (D) None of the above.

   (d) Which of the following statements concerning the sequence $a_n = \dfrac{n}{2n^2-3}$; $n = 1\,2\,3....$ is true:
   (A) Both $\{a_n\}$ and $\sum_{n=1}^{\infty} a_n$ are convergent.
   (B) $\{a_n\}$ is convergent but $\sum_{n=1}^{\infty} a_n$ is divergent.
   (C) Both $\{a_n\}$ and $\sum_{n=1}^{\infty} a_n$ are divergent.
   (D) None of the above.

   (e) Given function is continuous at $(0,0)$?
   $$f(x,y) = \begin{cases} \dfrac{\sin(xy)}{xy}, & \text{for } (x,y) \neq (0,0) \\ 0, & \text{for } (x,y) = (0,0) \end{cases}$$
   (A) $\lim_{(x,y)\to(0,0)} f(x,y)$ exists but the function is not continuous.
   (B) $\lim_{(x,y)\to(0,0)} f(x,y)$ exists and the function is continuous.
   (C) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist hence the function is not continuous.
   (D) None of the above.

(f) Consider two statements: (S_1) For a particle moving along a space curve, if the torsion (τ) is always zero, then the curve lies in a plane. (S_2) If a particle is moving with a constant speed, then its acceleration is zero.
(A) S_1 and S_2 both are true.
(B) S_1 is true and S_2 is false.
(C) S_2 is true and S_1 is false.
(D) None of the above.

(g) Let $P = (2, \pi/6)$ be a point in polar coordinates. Consider $P_1 = (-2, -5\pi/6)$, $P_2 = (2, 13\pi/6)$ and $P_3 = (2, -11\pi/6)$ in polar coordinates.
(A) P_1, P_2 and P_3 all represent P.
(B) P_1 and P_2 represent P but P_3 does not.
(C) P_1 and P_3 represent P but P_2 does not.
(D) None of the above.

2. (a) Define directional derivative of $z = f(x, y)$ in the direction $\hat{u}$ at a point (x_0, y_0) and explain it's geometric interpretation. Also write the formula for computing directional derivative for a differentiable function $f(x, y)$ using its gradient. **[2+3+1]**

(b) Let $f(x, y) = \begin{cases} 1, & xy = 0 \\ 0, & xy \neq 0 \end{cases}$, then compute $f_x(0, 0), f_y(0, 0)$ and directional derivative of f in the direction $\vec{u} = \hat{i} + \hat{j}$ at $(0, 0)$, if they exist. **[2+2+3]**

(c) Find the linearization of $f(x, y, z) = x^2 + xy + yz + 0.25z^2$ at $P(1, 1, 2)$. Then find an upper bound for the magnitude of the error in the standard linear approximation over the region D: $|x - 1| \leq 0.01; |y - 1| \leq 0.01; |z - 2| \leq 0.08$. **[2+3]**

3. (a) Consider the double integral

$$\int_{\sqrt{2}}^{2} \int_{\sqrt{4-y^2}}^{y} dx\, dy.$$

(i) Sketch the region R of integration in the xy-plane. **[2]**
(ii) Express the region using polar coordinates. **[2]**
(iii) Rewrite the integral using polar coordinates. **[1]**
(iv) Evaluate the integral in part (iii). **[3]**

(b) Sketch the region in the xy-plane bounded by the lines $y = -x + 1$, $y = x - 3$ and the curve $y = \sqrt{x - 1}$. Write an expression involving integrals in the order $dy\, dx$ that gives the area of this region. Then find the area. **[9]**

4. Let D be the region in the xyz-space defined by the inequalities

$$1 \leq y \leq 2, \quad 0 \leq xy \leq 2, \quad 0 \leq z \leq 2.$$

Evaluate the triple integral

$$\int\int\int_D (xy^2 + 3xyz)\, dx\, dy\, dz$$

by applying the transformation $u = y,\ v = xy,\ w = 3z$ and integrating over an appropriate region in the uvw-space. **[17]**

END of PART A

Birla Institute of Technology & Science, Pilani
First Semester 2023-2024, MATH F111 (Mathematics-I)
Comprehensive Examination (PART-B Open Book)

Max. Marks: 69 **December 18, 2023** **Time: 90 Min.**

---

1. Notations/symbols have their usual meaning.
2. Start answering each question on a new page, and answer the sub-parts of each question in continuation. Write **END** at the end of the last attempted question.

---

1. (a) Using polar coordinates and without using the double integral, compute the area of the region $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4,\ -1 \leq x \leq 1 \text{ and } y \geq 0\}$. **[8]**

   (b) Consider the space curve given by the following equation:

$$\mathbf{r}(t) = \sin(t)\,\mathbf{i} + (t + \cos t)\,\mathbf{j} + (t^3 - \pi t^2 + \frac{\pi^2}{4}t)\,\mathbf{k},\ \ t \neq 0 \quad \text{and} \quad \mathbf{r}(0) = 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k}.$$

   Find all the points t at which $\mathbf{r}(t)$ is smooth. Justify your answer with details. **[10]**

2. (a) Find a potential function f for the vector field $\mathbf{F} = (y^2z^2 + xe^{-2x})\mathbf{i} + (y\sqrt{y^2+1} + 2xyz^2)\mathbf{j} + (2xy^2z)\mathbf{k}$. **[10]**

   (b) Use Green's theorem to evaluate the integral $\oint_C \left(\frac{x}{x^2+1} - y\right) dx + \left(3x - 4\tan\frac{y}{2}\right) dy$, where C is the portion of $y = x^2$ from $(-1, 1)$ to $(1, 1)$, followed by the portion of $y = 2 - x^2$ from $(1, 1)$ to $(-1, 1)$. **[7]**

3. (a) Using Stokes' theorem, find the circulation of the vector field $\mathbf{F} = \left(-z + \frac{1}{2+x}\right)\mathbf{i} + (\tan^{-1} y)\mathbf{j} + \left(x + \frac{1}{4+z}\right)\mathbf{k}$ on the triangle C with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, traced counterclockwise when seen from the positive z-axis. **[12]**

   (b) Using divergence theorem, evaluate the outward flux of the vector field $\mathbf{F} = \left(x + e^{y\sin z}\right)\mathbf{i} + \left(\ln(\tan^{-1}(xz))\right)\mathbf{j} + \left(z + e^{\sqrt{x^2+y^2}}\right)\mathbf{k}$ across the boundary of $D = \{(x, y, z) \in \mathbb{R}^3 : 1 \leq x \leq 2, 1 \leq y \leq 2, 1 \leq z \leq 2\}$. **[5]**

4. (a) Find $\alpha \in \mathbb{R}$ for which $(1/4, 3/4)$ is the largest open interval of absolute convergence for the series $\sum_{n=1}^{\infty} (-1)^n \frac{2^{\alpha n}(2x-1)^{3n}}{n}$. For this α, does the series converge absolutely at the end points $1/4$ and $3/4$? Justify. **[11]**

   (b) Find the first three non-zero terms of the Taylor's series expansion of $f(x) = \ln(3 + \sin x)$ around π. **[6]**

END of PART B