Birla Institute of Technology & Science, Pilani (Raj.)

Second Semester 2022-2023, MATH F112 (Mathematics-II)

Mid-Semester Examination (Closed Book)

Time: 90 Minutes **Date: May 04, 2023 (Thursday)** **Max. Marks: 105**

1. There are total 3 questions. Start answering each question on a new page. Answer each sub-part of the questions in continuation. While answering, write all the steps with proper justification. Write "END" after the last attempted solution.
2. All notations used in the paper have the usual meaning. **Unless specified, the operations for any vector space are usual.**

Q 1. (a) Let $x_1, x_2, \ldots, x_n$ be solutions to the system $Ax = b$, where A is an $n \times n$ matrix and b is a non-zero vector in $\mathbb{R}^n$. Show that $a_1x_1 + a_2x_2 + \ldots + a_nx_n$ is a solution to $Ax = b$ if and only if $a_1 + a_2 + \ldots + a_n = 1$. **[12]**

(b) Either prove or disprove the following statements: **[7+8+3]**

(i) $V = \{f : \mathbb{N} \to \mathbb{R} \mid \exists\, n_f \in \mathbb{N} \text{ s.t. } f(n_f) = 0\}$ is a vector space.

(ii) $U = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is differentiable and } \dfrac{df}{dx} + 2f(x) = 0\}$ is a subspace of $F(-\infty, \infty)$.

(iii) $W = \{(x, y) \in \mathbb{R}^2 \mid x + 3y = 4,\ 2x - y = 4 \text{ and } 6x + 4y = 10\}$ is a subspace of $\mathbb{R}^2$.

Q 2. Let $T : P_2 \to \mathbb{R}^3$ be a mapping defined by

$$T(a + bx + cx^2) = (a, a + b + c, a + b - c).$$

Then answer the following questions:

(i) Prove that T is a linear transformation. **[6]**

(ii) Find $\ker(T)$, $R(T)$, a basis for $\ker(T)$, a basis for $R(T)$, dimension of $\ker(T)$ and dimension of $R(T)$. **[20]**

(iii) Is T one-to-one? Justify your answer. **[2]**

(iv) Is T onto? Justify your answer. **[2]**

(v) Does T^{-1} exist? If yes, find T^{-1}. **[8]**

Q 3. (a) For the vector space $V = (\mathbb{R}^3, \oplus, \odot)$ with operations

$$(x, y, z) \oplus (u, v, w) = (x + u - 2, y + v + 3, z + w - 1)$$
$$a \odot (x, y, z) = (ax - 2a + 2, ay + 3a - 3, az - a + 1), \quad a \in \mathbb{R},$$

prove that the set $S = \{(x, y, z) : x - 2y + z = 9\}$ is a subspace of V. Furthermore, prove that $B = \{(4, -2, 1), (2, -2, 3)\}$ is a basis of S. **[20]**

(b) Reduce the matrix $A = \begin{bmatrix} 2 & -4 & 2 & -2 \\ 2 & -4 & 3 & -4 \\ 4 & -8 & 3 & -2 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ into RREF and hence find the row space of A, the column space of A and the rank of A. **[17]**

END