Birla Institute of Technology & Science, Pilani
Second Semester 2022-2023, MATH F112 (Mathematics-II)
Comprehensive Examination

Max. Marks: 135 **July 10, 2023** **Time: 180 Min.**

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1. The question paper consists of two parts: **Part-A(Closed Book)** and **Part-B(Open Book). Part-A** is of 45 Marks & 60 Minutes and **Part-B** is of 90 Marks & 120 Minutes. Attempt questions of **Part-A** and **Part-B** on two separate answer sheets.
2. Answer sheet of Part-B will be given only after submitting the Part-A answer sheet. Part-A can be submitted anytime between **8:45 AM** and **9:15 AM** . Write Part-A/Part-B on the top right corner of the respective answer sheet.
3. Notations/symbols have their usual meaning.
4. Start answering each question on a new page and answer the sub-parts of each question in continuation. Write **END** at the end of the last attempted question in each answer sheet.
5. The use of a calculator is not allowed.

Max. Marks: 45 **PART-A(Closed Book)** **Time: 60 Min.**

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1. (a) Let $B = \{-x^2 + 2x + 1, x + 1, -2x^2 + 2x + 1\}$ and $B' = \{-x^2 + x, x, x + 1\}$ be the ordered bases for P_2. If $[v]_B = [2, 0, 1]$ for some $v \in P_2$, then using $[v]_B$ determine $[v]_{B'}$. Also, find v. **[10]**

   (b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator defined as

   $$T(x, y, z) = (x + y + z, 2y + z, 2y + 3z).$$

   If A is the matrix of T relative to the standard basis of $\mathbb{R}^3$, then find A and all the eigenvalues of A. Further, find a basis for the eigenspace corresponding to the smallest eigenvalue of A. **[12]**

2. (a) Find all the values of $\sin^{-1}(-i)$. **[9]**

   (b) Find the analytic function $f(z) = u(r, \theta) + iv(r, \theta)$ such that $f(1 + i\sqrt{3}) = i(1 + 3\sqrt{3})$, where $u(r, \theta) = r^2 \sin 2\theta + r \sin\theta - 3\sqrt{3}$. Also, express f in terms of z. **[11]**

   (c) Let

   $$f(z) = \begin{cases} \dfrac{z^5}{|z|^4} & z \neq 0, \\ 0 & z = 0. \end{cases}$$

   Is f differentiable at $z = 0$? Justify your answer. **[3]**

END of PART A

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Comprehensive Examination (PART-B Open Book)

Max. Marks: 90 | **July 10, 2023** | **Time: 120 Min.**

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1. Notations/symbols have their usual meaning.
2. Start answering each question on a new page, and answer the sub-parts of each question in continuation. Write **END** at the end of the last attempted question.

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1. (a) Let V and W be vector spaces and $T : V \to W$ be a linear transformation. If $v_1, v_2, \ldots, v_n$ be vectors in V, then prove or disprove the following:
   (i) If the set $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is linearly independent in W, then $\{v_1, v_2, \ldots, v_n\}$ is linearly independent in V. **[3]**
   (ii) If the set $\{v_1, v_2, \ldots, v_n\}$ is linearly independent in V, then $\{T(v_1), T(v_2), \ldots, T(v_n)\}$ is linearly independent in W. **[3]**

   (b) Let $T : P_2 \to P_2$ be a linear transformation such that

$$T(1+x) = 1 + x^2, \quad T(x + x^2) = x - x^2, \quad T(1 + x^2) = 1 + x + x^2.$$

   Find $T(a + bx + cx^2)$. **[9]**

   (c) Evaluate $\displaystyle\int_C \bar{z}e^z dz$, where C is the positively oriented square with vertices 0, 1, $1+i$, and i. **[15]**

2. (a) Without using residues, evaluate

$$\int_C \frac{\mathrm{Log}(z+3)}{(z-1)(z-2)^2} dz,$$

   where $C : |z - 3| = 5/2$, is traversed in the counterclockwise direction. **[15]**

   (b) Determine all the entire functions f such that $\mathrm{Im}(f(z)) \geq \mathrm{Im}(z)$ for all $z \in \mathbb{C}$. **[12]**

3. (a) Find the Laurent series expansions of

$$f(z) = \frac{z^2 + z + 3}{z^3 + 2z^2 + z + 2}$$

   about the origin indicating all the possible regions where these expansions are valid. **[11]**

   (b) Find and classify all the singularities (essential, pole, removable etc.) of

$$f(z) = \frac{e^z - 1}{\sin z}$$

   and use residues to evaluate the integral $\int_C f(z) dz$, where C is the positively oriented circle $|z - \frac{3\pi}{2}| = \pi$. **[12]**

   (c) Using the Cauchy residue theorem, evaluate

$$\int_0^{2\pi} \frac{\cos\theta}{2 + \cos\theta} d\theta. \quad \textbf{[10]}$$

END of PART B