Birla Institute of Technology & Science, Pilani

Second Semester 2022-2023, MATH F113 (Probability & Statistics)
Comprehensive Examination-Part A (Open book)

Time: 110 Min. **Date: July 8, 2023 (Saturday)** **Max. Marks: 80**

1. There are two parts A and B. Part A has 7 questions and Part B has 4 questions. Write the answers of Part A and Part B in separate answer sheets provided and write respective parts in the top left corner of the cover page.
2. Start answering each question on a new page. Answer each sub-part of the questions in continuation. While answering, write all the steps with proper justification. Just writing the correct answer will receive no credit.
3. Define events/random variables (with distribution and parameters) as and when required.
4. Notations/symbols have their usual meaning.

1. A box contains three coins: two fair coins and one with heads on both sides.
   (i) A coin is picked at random and tossed once. Find the probability of getting the head? **[6]**
   (ii) A coin is picked at random and tossed once. If the head is observed, what is the probability that the two-headed coin was chosen? **[4]**

2. (i) In an examination, a student appears and answers all 50 multiple-choice questions. Answer to each question has 4 possible options with one correct option. He knows the answer to 10 questions correctly, but has no idea about the other 40 questions for which he chooses options randomly. Let X be the total number of correct answers. Find the pmf of X. Also, find the mean of X. **[12]**
   (ii) Let X and Y be two independent random variables. Suppose that $V(2X - Y) = 6$ and $V(X + 2Y) = 9$. Find $V(X)$ and $V(Y)$. **[4]**

3. Suppose that the customers arrive at a store in Poisson process at the average rate of 1/10 per minute.
   (i) Find the density function of the total time (T) in minutes before the arrival of the fourth customer. **[3]**
   (ii) Find the mean and standard deviation of T. **[4]**
   (iii) Find the probability that T is less than 15 **[6]**

4. The students' data of average classes attended in a month and their total marks in a test of 60 marks are as follows:

x (Attendance)	2	4	6	8	10	12	15	20	24	30
y (marks)	7	13	15	19	25	28	36	38	42	55

   (i) Find the estimated regression line of y on x. **[8]**
   (ii) Find the point estimate of variance and the coefficient of determination. **[6]**

5. Let X and Y be two continuous random variables whose joint pdf f_{XY} is given by:

$$f_{XY}(x,y) = \begin{cases} kxy^2; & \text{if } x \geq 0,\ y \geq 0,\ 0 \leq x+y \leq 1, \\ 0; & \text{otherwise,} \end{cases}$$

where k is the constant.

(i) Find the value of k. **[2]**

(ii) Find the marginal pdf of X and the marginal pdf of Y. **[6]**

(iii) Are X and Y independent? Justify your answer. **[2]**

(iv) Find the conditional expectation of Y given that $X = \frac{1}{2}$. **[5]**

6. The monthly demand for Titan watches at a particular store is known to have the following probability distribution:

Demand	1	2	3	4
Probability	0.2	0.3	0.4	0.1

(i) Assume that the demand in each month is independent. Let X_1 and X_2 denote the demands for watches in September and November respectively. Find the pmf of $T = \frac{1}{2}(X_1 + X_2)$. **[4]**

(ii) Hence, compute the expectation of T and find $P(T > 2.5)$. **[4]**

7. Let X_1 and X_2 be independent Poisson random variables with parameters μ_1 and μ_2 respectively. Let $Y = X_1 + X_2$. Find the distribution of Y. **[4]**

END

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Second Semester 2022-2023, MATH F113 (Probability & Statistics)

Comprehensive Examination-Part B (Open Book)

Time: 70 Min. Date: July 8, 2023 (Saturday) Max. Marks: 55

1. There are total 4 questions in this part. Start answering each question on a new page. Answer each sub-part of the questions in continuation. While answering, write all the steps with proper justification. Just writing the correct answer will receive no credit.
2. Define events/random variables (with distribution and parameters) as and when required.
3. Notations/symbols have their usual meaning.

1. Let X be a normal random variable with unknown mean μ and unknown variance σ^2. Consider two independent random samples on X of size 10 and 15 with respective sample variances S_1^2 and S_2^2, and let S^2 denote the sample variance of the sample of size 25 obtained by mixing these two samples.
   (i) Determine if $\widehat{\sigma^2} = (9S_1^2 + 14S_2^2)/23$ is an unbiased estimator of σ^2. **[3]**
   (ii) Find the preferable estimator of σ^2 among $\widehat{\sigma^2}$ and S^2, by comparing their variances. **[10]**

2. (i) A pollster states that 4 is the left endpoint of a 95% (2-sided) confidence interval for the standard deviation of a normally distributed random variable, based on the sample of size 31. Compute the value of the sample standard deviation obtained in the sample based on the given information, and hence estimate the right endpoint of this (2-sided) confidence interval. **[7]**
   (ii) For a random sample $X_1, X_2, \ldots, X_n$ on X, find the maximum likelihood estimators of parameters α and λ of the following distribution

$$f(x; \alpha, \lambda) = \frac{1}{\Gamma(\lambda)} \left(\frac{\lambda}{\alpha}\right)^{\lambda} e^{-\frac{\lambda x}{\alpha}} (x)^{\lambda - 1}; \quad 0 \le x < \infty, \quad \lambda, \alpha > 0.$$

   Use if required: $\frac{\partial}{\partial \lambda} \ln \Gamma(\lambda) = \ln \lambda - \frac{1}{2\lambda}$. **[8]**

3. A car manufacturer is conducting a study to evaluate the stopping distance (in feet) of a new model of the car under specific experimental conditions. The company sets a maximum allowable stopping distance of 150 feet. A random sample of 30 cars is tested, and the stopping distances are recorded. The sample mean stopping distance is 155 feet. It is known that the stopping distances follow a normal distribution with a population standard deviation of 10 feet. **[15]**
   (i) Does the data suggest that the true average stopping distance exceeds the maximum allowable value? Formulate the appropriate hypotheses and perform the test at $\alpha = 0.05$.
   (ii) Determine the probability of the Type II error when $\alpha = 0.05$ and the actual value of the average stopping distance μ is 160 feet.
   (iii) What sample size would be necessary to have a power of 0.80, when $\alpha = 0.05$ and $\mu = 160$?

4. A coffee shop owner claims that more than 70% of their customers prefer their new signature blend. To challenge this claim, a random sample of 80 customers is taken, and it is found that 54 of them prefer the new blend. Using a $P-$value approach, test the hypotheses at $\alpha = 0.06$?
   . **[12]**

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