Birla Institute of Technology & Science, Pilani

MATH F211 (Mathematics-III)
First Semester 2023-2024
Comprehensive Examination: PART-A (Closed Book)

Time: 100 Minutes **Date: December 14, 2023 (Thursday)** **Max. Marks: 75**

1. Notations and symbols have their usual meaning.
2. Start a new question on a fresh page. **Moreover, answer each subpart of a question in continuation.**
3. Write **END** at the end of the last attempted question.

Q.1 (a) Using the method of Laplace transforms, solve the following initial value problem:

$$y'' + 2y' + 5y = 3e^{-x}\cos x,\ y(0) = 0,\ y'(0) = 0.$$ **[8]**

(b) Using the method of Laplace transforms, evaluate the following integrals:

(i) $\int_0^\infty e^{-x}x^2 \sin x\ dx$ **(ii)** $\int_0^\infty \frac{\sin(xt)}{t}\ dt,\ t > 0$ **[7 + 4]**

Q.2 Using the method of variation of parameters, find the general solution of the following system of differential equations

$$\frac{dx}{dt} = 7x + 6y + e^{-t}$$
$$\frac{dy}{dt} = 2x + 6y + 2e^{-t}$$

without converting into a second order differential equation. **[18]**

Q.3 Find the Fourier series of $f(x)$, where

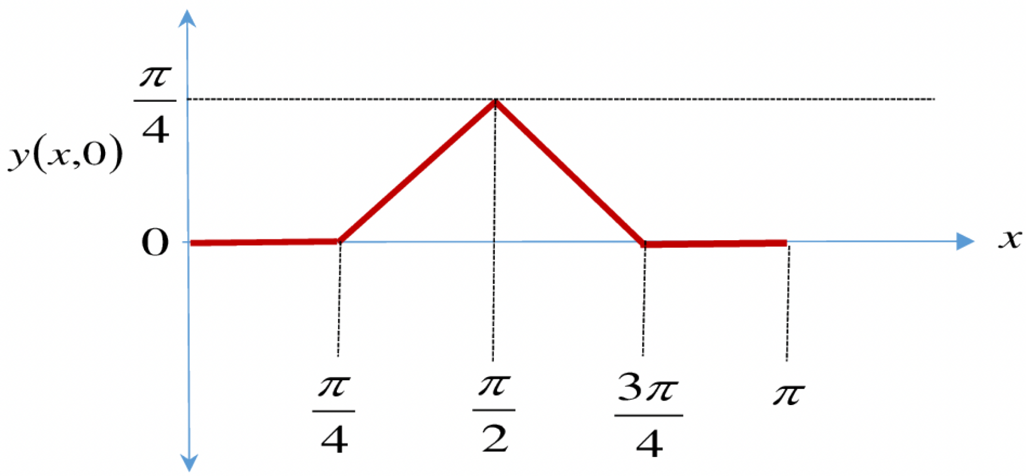
$$f(x) = \begin{cases} x^2, & -\pi \le x \le 0 \\ 0, & 0 < x < \pi. \end{cases}$$

Hence, evaluate the sum of infinite series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$ **[16]**

Q.4 (a) The one-dimensional equation for the vibrating string of length $L > 0$ which is fixed at the two end points ($x = 0$ and $x = L$) can be given as $a^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$, where $a > 0$. Now, by using the method of the separation of variables, compute the expression for the general solution of the deflection $y(x,t)$ such that the initial transverse velocity $\frac{\partial y}{\partial t}(x,0)$ of the string is zero and the initial spatial profile $y(x,0)$ is $f(x)$. **[14]**

(b) Now, by using the results obtained in (a) compute the deflection $y(x, t)$ (first three terms) of a vibrating string of length $L = \pi$ with its ends ($x = 0$ and $x = \pi$) fixed and zero initial transverse velocity. The initial spatial profile $f(x)$ of the string is given as: **[8]**



END

Birla Institute of Technology & Science, Pilani

MATH F211 (Mathematics-III)
First Semester 2023-2024
Comprehensive Examination: PART-B (Open Book)

Time: 80 Minutes **Date: December 14, 2023 (Thursday)** **Max. Marks: 60**

1. Notations and symbols have their usual meaning.
2. Start a new question on a fresh page. **Moreover, answer each subpart of a question in continuation.**
3. Write **END** at the end of the last attempted question.

Q.1 Find the solution of the initial value problem

$$(2x\log x)^2 y'' + 4x(\log x)^2 y' + y = \log x \;\; (x > e) \;\text{ such that }\; y(e) = 1, y'(e) = 0.$$ **[16]**

Q.2 (a) Prove or disprove that

$$\sum_{m=0}^{n} (2m+1)P_m(x)P_m(y) = \frac{n+1}{x-y}\{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)\}, \text{where } x \neq y.$$

Hence, find the value of $\sum_{m=0}^{10} (2m+1)P_m(x)$ in terms of P_{10} and P_{11}. **[12]**

(b) Evaluate the integral

$$I = \int_{-1}^{1} \frac{1}{\sqrt{1-x}} P_n(x)dx.$$ **[10]**

Q.3 (a) If $\int \frac{J_1(\sqrt{x})}{\sqrt{x}} \sin(\sqrt{x})\, dx = A(x)\cos(\sqrt{x}) + B(x)\sin(\sqrt{x})$, where $x > 0$, then find $A(x)$ and $B(x)$. **[12]**

(b) Using mathematical induction, prove that

$$\left(\frac{1}{x}\frac{d}{dx}\right)^m [x^{-n}J_n(x)] = (-1)^m \frac{1}{x^{n+m}} J_{n+m}(x), \qquad x > 0$$

where $\left(\frac{1}{x}\frac{d}{dx}\right)^m y = \left(\frac{1}{x}\frac{d}{dx}\right)^{m-1}\left(\frac{1}{x}\frac{dy}{dx}\right)$ for all $m \in \mathbb{N}$ **[10]**

END