Birla Institute of Technology & Science, Pilani

First Semester 2022-2023

MATH F214 (Elementary Real Analysis)

Comprehensive Examination Part A (Closed Book)

Time: 120 Min. **Date: December 21, 2022 (Wednesday)** **Max. Marks: 55**

1. Write PART-A on the top of your answer sheet.
2. While answering, justify your steps. Just writing the final answer will receive no credit.
3. Write **END** after the last attempted solution.
4. You can submit PART-A any time before 11 : 15 AM and start Part-B. Ideally you should not spend more than 120 minutes on part-A.

1. Let $\mathbb{R}$ be a metric space with respect to the Euclidean metric and $\mathbb{I} \subseteq \mathbb{R}$. Find all the interior points of $\mathbb{I}$. (Here $\mathbb{I}$ denotes the set of irrational numbers) **[3]**

2. Let (X, d) be a metric space and $\{G_\alpha\}$ be any collection of subsets of X.

   (a) If G_α is open for each α, then show that $\bigcup_\alpha G_\alpha$ is open in X. **[4]**

   (b) If G_α is closed for each α, then either prove or disprove that $\bigcup_\alpha G_\alpha$ is closed. **[6]**

3. Let d be the Euclidean metric or $\mathbb{R}$ and d_{disc} be the discrete metric on $\mathbb{R}$. Let $f : (\mathbb{R}, d_{disc}) \to (\mathbb{R}, d)$ be defined as
$$f(x) = x, \ \forall x \in \mathbb{R}.$$
   Either prove or disprove that f is continuous. **[4]**

4. Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that if $f(a) < c < f(b)$, then there exists $x_0 \in (a, b)$ such that $f(x_0) = c$. **[10]**

5. Let $g : \mathbb{R} \to \mathbb{R}$ be a non-zero function such that
$$g(x + y) = g(x)g(y).$$
   Show that if g is continuous at $x = 0$, then g is continuous at every $x \in \mathbb{R}$. **[7]**

6. Show that a bounded function f on $[a, b]$ is integrable if and only if for each $\epsilon > 0$ there exists a partition P_ϵ of $[a, b]$ such that
$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$ **[8]**

7. Let f be a continuous function on $\mathbb{R}$ and define
$$G(x) = \int_0^{\sin x} f(t)dt, \ x \in \mathbb{R}.$$
   Show that G is differentiable on $\mathbb{R}$ and compute G'. **[5]**

8. Show that $f_n(x) = \dfrac{nx}{1 + n^3x^2}$, $x \in [0, 1]$ converges uniformly on $[0, 1]$. **[8]**

End of Paper