Birla Institute of Technology and Science, Pilani (Pilani Campus)

Elementary Real Analysis (MATH F214)

Mid-Semester Examination

Max. Marks: 60 **November 01, 2022** **Time: 90 Minutes**

Note: (i) Except for the Q 1, justify your answers and write your arguments clearly. If a results from lectures/textbook is used, it must be stated clearly.
(ii) $\mathbb{Q}, \mathbb{I}$, and $\mathbb{R}$ denote the set of rational numbers, the set of irrational numbers and the set of real numbers, respectively.

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1. State(without justification) whether the following statements are true or false: **[10]**

   (i) If A is a nonempty, bounded subset of $\mathbb{Q}$, then $\sup(A) \in \mathbb{Q}$.

   (ii) There exists an infinite subset of $\mathbb{R}$ with no limit points.

   (iii) For every $x \in \mathbb{I}$ there exists a sequence of real numbers (x_n) converging to x.

   (iv) Every finite subset of $\mathbb{R}$ is compact.

   (v) The set of all binary sequences is countably infinite.

   (vi) The series $\sum \frac{(-1)^n}{n}$ is absolutely convergent.

   (vii) Every open cover of $\mathbb{R}$ has a finite subcover.

   (viii) Every closed subset of $\mathbb{R}$ is compact.

   (ix) Every bounded sequence is convergent.

   (x) The set of interior points of $\mathbb{I}$ is the empty set.

2. Show that $\inf\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 0$. **[3]**

3. Let A be a nonempty, bounded subset of $\mathbb{R}$ and $\alpha = \inf(A)$. Show that there exists a sequence (a_n) in A such that $a_n \to \alpha$. **[6]**

4. Determine if the following series is convergent or divergent: **[5]**

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}.$$

5. Either prove or disprove that the set of all functions $f : \mathbb{N} \to \{0, 1\}$ is countably infinite. **[6]**

6. Show that every contractive sequence of real numbers is convergent. Further, show that converse is not true in general by producing a counterexample. **[10]**

7. Let (X, d) be a metric space. Show that

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

   is a metric on X. **[12]**

8. Let (X, d) be a metric space and $\{K_\alpha\}$ be a collection of compact subsets of X. If intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then show that $\bigcap_\alpha K_\alpha$ is nonempty. **[8]**