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Comprehensive Examination
MATH F214 Elementary Real Analysis

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Answer all the questions.

(1) Let (a_n) be a sequence of real numbers. Justify the following statements:
  (a) If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.
  (b) The converse in (a) need not be true.
  (c) If the series $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$.
  (d) The converse in (c) need not be true. **(Marks: 4x3=12)**

(2) State whether each of the following metric space (X, d) is complete or not, and justify your answer:
  (a) $X = \mathbb{R} \setminus \mathbb{Q}$ with $d(x, y) = |x - y|$ for $x, y \in X$.
  (b) X is a non-empty set and d is the discrete metric.
  (c) $X = (0, 1]$ with $d(x, y) = |x - y|$ for $x, y \in X$.
  (d) $X = \{\frac{n}{n+1} : n \in \mathbb{N}\}$ with $d(x, y) = |x - y|$ for $x, y \in X$. **(Marks: 4x2= 8)**

(3) Let (X, d) be a metric space and let (x_n) and (y_n) be a Cauchy sequences in X. For $n \in \mathbb{N}$, let $a_n = d(x_n, y_n)$. Show that the sequence (a_n) is a convergent sequence in $\mathbb{R}$. **(Marks: 4)**

(4) Let (X, d) be a metric space and $A \subseteq X$.
  (a) What is interior of A.
  (b) What is the condition under which A is said to be dense in X?
  (c) Prove that A is dense in X iff interior of A^c is empty. **(Marks: 2+2+4= 8)**

(5) State whether each of the following metric space (X, d) is compact or not, and justify your answer:
  (a) $X = [0, 1]$ with $d(x, y) = |x - y|$ for $x, y \in X$.
  (b) $X = [0, 1]$ and d is the discrete metric.
  (c) $X = (0, 1]$ with $d(x, y) = |x - y|$ for $x, y \in X$.
  (d) $X = \{1\} \cup \{\frac{n}{n+1} : n \in \mathbb{N}\}$ with $d(x, y) = |x - y|$ for $x, y \in X$. **(Marks: 4x2= 8)**

(6) Show that every compact metric space is complete.
Is the converse true? Justify your answer. **(Marks: 6)**

(7) Let (X, d) be a metric space. Prove that X is disconnected iff there exists a continuous onto function $f : X \to \{0, 1\}$, where $Y := \{0, 1\}$ is with discrete metric. **(Marks: 6)**

(8) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Prove the following:
  (a) If f is Riemann integrable, then $g : [a, b] \to \mathbb{R}$ defined by and $g(x) = \int_a^x f(t)\, dt$ for $x \in [a, b]$, is continuous.
  (b) If $f : [a, b] \to \mathbb{R}$ is continuous, then g as in (a) is differentiable and $g'(x) = f(x)$ for every $x \in [a, b]$. **(Marks: 4+4 =8)**

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