Birla Institute of Technology and Science, Pilani
Department of Mathematics
MATH F215; Algebra 1
First Semester 2017-2018
Date: 3 December 2017

Time: **90** Minutes — **Comprehensive Exam [Open Book]** — Max. Marks: **40**

Instructions:

- This question paper has **SEVEN** questions. Answer **ALL** questions.
- Answer of all the subpart of a question must be written together.
- Answer without proper justification will attract zero mark.
- Direct use of unsolved exercise from the text book without proof is **NOT** permissible.

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1. Prove or disprove the following:
   (i) The intersection of two distinct maximal ideals of a ring R is a maximal ideal of R. **[4]**
   (ii) For the ring $R = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, the set $S = \{(a, b, c) \in R \mid a + b = c\}$ is a subring of R. **[3]**
2. Find the greatest common divisor of $x^3 - 2x + 1$ and $x^2 - x - 2$ in $\mathbb{Z}_5[x]$, and express it as a linear combination of the given polynomials. **[5]**
3. Let R be a Euclidean domain. Prove that
   (i) $d(a) = d(-a)$ for all $a \neq 0 \in R$. **[4]**
   (ii) If $d(a) = 0$ for $a \neq 0 \in R$, then a is unit in R. **[3]**
4. Let R be a ring with unity. Using its elements, we define a ring R' by defining
   $$a \oplus b = a + b + 1 \text{ and } a \otimes b = ab + a + b,$$
   where $a, b \in R$ and the addition and multiplication on the right hand side of these relation are those of R. Prove that R is isomorphic to R'. **[8]**
5. Find all the ring homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$. **[5]**
6. Prove that any nonzero ideal in the ring $\mathbb{Z}[i] = \{m + ni \mid m, n \in \mathbb{Z}\}$ of Gaussian integers must contain some positive integer. **[4]**
7. Let N and M be two normal subgroups of a group G such that $N \cap M = \{e\}$. Show that for any $n \in N, m \in M$, $nm = mn$. **[4]**

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Time: **90** Minutes | **Comprehensive Exam [Closed Book]** | Max. Marks: **40**

Instructions:

- This question paper has **Five** questions. Answer **ALL** questions.
- Answer of all the subpart of a question must be written together.
- Answer without proper justification will attract zero mark.

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1. Prove or disprove the following:

   (i) The rings $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are isomorphic. **[4]**

   (ii) Subring of a non-commutative ring is non-commutative. **[4]**

   (iii) Let $\langle x^4 + 1 \rangle = \{(x^4 + 1)f(x) \mid f(x) \in \mathbb{Q}[x]\}$ be an ideal of the polynomial ring $\mathbb{Q}[x]$. Then the quotient ring $\frac{\mathbb{Q}[x]}{\langle x^4+1 \rangle}$ is a field. **[5]**

   (iv) The mapping $f : \mathbb{Z}_2 \to \mathbb{Z}_2$ defined by $f(n) = n^2 - n$ for all $n \in \mathbb{Z}_2$ is a ring homomorphism. **[4]**

2. Define unique factorization domain. Show that the ring $\mathbb{Z}[\sqrt{-7}\ ]$ is not a unique factorization domain. **[7]**

3. Define Euclidean domain. Show that every field F is a Euclidean domain. **[4]**

4. For an integer n_0, define $\langle n_0 \rangle = \{n_0 a \mid a \in \mathbb{Z}\}$. Prove that $M = \langle n_0 \rangle$ is a maximal ideal of the ring $\mathbb{Z}$ of integers if and only if n_0 is a prime number. **[8]**

5. Let $\sigma = (1\ 2\ 3)(1\ 2\ 5\ 6) \in S_6$. Find the order of σ and then determine σ^{35}. **[4]**

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