Birla Institute of Technology and Science, Pilani
MATH F215 (Algebra 1)
First Semester 2022-2023
Mid-Semester Examination [Closed Book]

Max. Marks: 70 **Max. Time: 90 Minutes** **Date: Nov. 02, 2022**

- Support your conclusions always with proper explanations.

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1. Prove or disprove the following:

   (i) Every group of order 14 contains 6 elements of order 7. **[5]**

   (ii) Let $a, b \in G$ such that $ab = ba$ and $(o(a), o(b)) = 1$. Then $o(ab) = o(a)o(b)$. **[6]**

   (iii) The number of generators of the group $\mathbb{Z}_{2022}$ is 670. **[3]**

   (iv) There exists a non-abelian group of order 125 with trivial center. **[3]**

   (v) Let G be a group such that $o(G) = 9$. Then the class equation of G is $9 = 1 + 2 + 3 + 3$. **[3]**

2. Consider a map $f : 4\mathbb{Z} \to \mathbb{Z}_3$ defined by $f(4n) = [n]$ for all $n \in \mathbb{Z}$. Prove that f is an onto homomorphism. Hence, deduce that the groups $\frac{4\mathbb{Z}}{12\mathbb{Z}}$ and $\mathbb{Z}_3$ are isomorphic. **[6]**

3. Determine all the group homomorphisms from $\mathbb{Z}_9 \longrightarrow U_9$. Write all these homomorphisms explicitly. **[10]**

4. State and prove Sylow's second theorem. **[10]**

5. Prove that any group of order 30 is not simple. **[6]**

6. Find $\ker(\phi)$ and $\phi(20)$ for the homomorphism $\phi : \mathbb{Z} \longrightarrow S_{2022}$ such that

$$\phi(1) = (1\ 4\ 2\ 6)(2\ 5\ 7).$$ **[6]**

7. Let $o(G) = pq$, where p and q are distinct primes such that $p < q$. Show that if p does not divide $(q - 1)$, then G is cyclic. **[12]**

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