Birla Institute of Technology and Science, Pilani
Department of Mathematics
MATH F215; Algebra 1
First Semester 2022-2023
Date: 23 December 2022

Time: **90** Minutes | **Comprehensive Exam [Closed Book]** | Max. Marks: **45**

Instructions:

- This question paper has **Fifteen** multiple choice questions with only one correct answer. Each correct answer carries 3 marks and wrong one gets **-1**.
- Write your answers only in the given boxes.
- **Any over-writing in your answer will be considered as unattempted and will carry zero mark.**

Name: **ID:**

Signature of Invigilator:

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.															

1. The cardinality of the center of the group $\mathbb{Z}_{12}$ is

(A) 1 (B) 2 (C) 4 (D) 12

2. The number of 5-Sylow subgroups of the group $\mathbb{Z}_{20}$ is

(A) 1 (B) 2 (C) 3 (D) 4

3. The number of all the automorphisms of the group $(\mathbb{Z}, +)$ is

(A) 1 (B) 2 (C) 3 (D) 4

4. Consider the ring $R = M_2(\mathbb{Z})$ of all 2×2 matrices with integer entries and the subset $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{Z} \right\}$ of R. Then which of the following statement is correct:

(A) S is a subring of R but not an ideal of R

(B) S is an ideal of R

(C) S is neither a subring nor an ideal of R

(D) S is an ideal of R but not a subring of R

5. The number of maximal ideals of the ring $\mathbb{Z}_8$ is

(A) 1 (B) 2 (C) 3 (D) 4

6. Let I denote the set of all polynomials in $\mathbb{Z}[x]$ with constant term zero. Then which of the following is correct:

(A) I is not an ideal of $\mathbb{Z}[x]$

(B) I is an ideal of $\mathbb{Z}[x]$ but not a maximal ideal of $\mathbb{Z}[x]$

(C) I is a maximal ideal of $\mathbb{Z}[x]$

(D) I is not a subring of $\mathbb{Z}[x]$

7. Which one of the following statement is not true?

(A) $\mathbb{Z}[\sqrt{3}]$ is a Euclidean domain

(B) $\mathbb{Z}[\sqrt{3}]$ is a principal ideal domain

(C) $\mathbb{Z}[x]$ is a Euclidean domain

(D) $\mathbb{Z}_7$ is a Euclidean domain

8. Which one of the following statement is not true?

(A) $\mathbb{Z}[\sqrt{-3}]$ is not a unique factorization domain

(B) $\mathbb{Z}[\sqrt{-6}]$ is not a unique factorization domain

(C) $\mathbb{Z}_6$ is a unique factorization domain

(D) $\mathbb{Z}_7$ is a unique factorization domain

9. The value(s) of $c \in \mathbb{Z}_3$, such that $\frac{\mathbb{Z}_3[x]}{\langle x^3+cx^2+[1]\rangle}$ is a field, must be

(A) both [0] and [1] (B) [1] only (C) [2] only (D) both [0] and [2]

10. Consider the polynomials $f(x) = x^3 - 6x + 2$ and $g(x) = x^3 + x^2 - 2x - 1$. Then

(A) $f(x)$ is irreducible over $\mathbb{Q}$ but $g(x)$ is reducible over $\mathbb{Q}$

(B) $f(x)$ is reducible over $\mathbb{Q}$ but $g(x)$ is irreducible over $\mathbb{Q}$

(C) both $f(x)$ and $g(x)$ are irreducible over $\mathbb{Q}$

(D) Neither $f(x)$ nor $g(x)$ is irreducible over $\mathbb{Q}$

11. Which of the following can not be an order of a finite field?

(A) 3 (B) 5 (C) 16 (D) 18

12. Which one of the following is a subring of $\mathbb{Z}[i]$?

(A) $\{a + bi \in \mathbb{Z}[i] \ : \ a \text{ and } b \text{ are even integers}\}$

(B) $\{a + bi \in \mathbb{Z}[i] \ : \ a \geq 0\}$

(C) $\{a + bi \in \mathbb{Z}[i] \ : \ b \geq 0\}$

(D) $\{a + bi \in \mathbb{Z}[i] \ : \ a, b \geq 0\}$

13. The characteristic of the quotient ring $\frac{\mathbb{Z}[i]}{\langle 3+i\rangle}$ is equal to

(A) 1 (B) 5 (C) 10 (D) 15

14. The number of associates of $8 + 3i$ in the ring $\mathbb{Z}[i]$ is

(A) 1 (B) 2 (C) 3 (D) 4

15. Which one of the following statement is true?

(A) Integral domain is a field

(B) Every finite integral domain is not a Euclidean domain

(C) Finite integral domain is a field

(D) Divison ring is a field

****END****

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Instructions:

- This question paper has **SIX** questions. Answer **ALL** questions.
- Answer of all the subpart of a question must be written together.
- Answer without proper justification will attract zero mark.
- Direct use of unsolved exercises from the text book without its solution is **NOT** permissible.

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1. Prove or disprove the following:
   (i) The fields $\mathbb{R}$ and $\mathbb{C}$ are isomorphic. **[5]**
   (ii) Any homomorphism of a field into a ring R is either one-one or it takes every element to zero. **[5]**
   (iii) The set $I = \{a + b\sqrt{3} \in \mathbb{Z}[\sqrt{3}] \mid a - b \text{ is an even integer}\}$ is an ideal of the ring $\mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\}$. **[5]**
2. Use the Euclidean algorithm to find $\gcd(x^8 - 1, x^6 - 1)$ in $\mathbb{Q}[x]$ and write the obtained gcd as a linear combination of $x^8 - 1$ and $x^6 - 1$. **[5]**
3. Find an irreducible polynomial of degree 2 over the ring $\mathbb{Z}_5$. Use it to construct a field of 25 elements. Justify your answer. **[5]**
4. Using fundamental theorem of ring homomorphism, prove that the rings $\mathbb{Z}/5\mathbb{Z}$ and $\mathbb{Z}_5$ are isomorphic. **[10]**
5. Let $(G, +)$ be a simple commutative group and R be the set of all homomorphisms from G to G. Then R forms a ring under the operations defined as
   $$(f \oplus g)(x) = f(x) + g(x) \;\; ; \;\; (f \circ g)(x) = f(g(x)) \text{ for all } f, g \in R \text{ and } x \in G$$
   Prove that $(R, \oplus, \circ)$ is a division ring. [**5**]
6. Let G be an infinite cyclic group. Then prove that G has exactly two generators which are inverses of each other. **[5]**

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