Birla Institute of Technology and Science, Pilani (Pilani Campus)
First Semester 2023-24

Course Number: MATH F215 (Algebra-I)	Examination: Comprehensive Exam (Part-A)
Date of Examination: 17.12.2023	Maximum Duration: 90 min
Maximum Marks: 45	Mode: Closed Book

Instructions.

1. Notation used is as defined in the lectures and tutorials.
2. **For Question 1: write the answer at the back of the first sheet of the answer booklet only – otherwise it will not be graded. Answer all its parts consecutively. For each part, there is only one correct option. In case of overwriting, no recheck requests will be entertained.**
3. Justify all answers. You will be graded on the correctness of your solution as well as the quality of your explanation. Answers without justification will lead to a score of zero. Illegible answers will not be graded.
4. If multiple answers are written for the same question, only the first one will be graded.
5. **Answer all the parts of a question consecutively. Write END at the end of your answers.**
6. Students found cheating will be awarded a score of zero and will be reported to the concerned authorities.

Questions.

Question 1. **[10]**

(i) Select the **False** statement.

(A) If G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$.

(B) If G is a finite group and d is a positive integer such that d divides $|G|$, then G has a subgroup of order d.

(C) If G is a group and $|G| = p$, where p is a prime, then G has an element of order p.

(D) None of the above.

(ii) Select the **False** statement.

(A) The set $\{2^m : m \in \mathbb{Z}\}$ is a subgroup of $(\mathbb{R} \setminus \{0\}, \cdot)$.

(B) If G is a group with operation $*$, then a non-empty finite subset H of G is a subgroup of G if and only if H is closed under $*$.

(C) $(\mathbb{Q}, +)$ is a cyclic group.

(D) None of the above.

(iii) Select the **False** statement.

(A) If N is a normal subgroup of G, then $gn = ng$ for all $n \in N$, $g \in G$.

(B) If N is a normal subgroup of G, then $gng^{-1} \in N$ for all $n \in N$, $g \in G$.

(C) If N is a normal subgroup of G, then $gN = Ng$ for all $g \in G$.

(D) None of the above.

(iv) Select the **False** statement.

(A) If R is a ring and $a, b \in R$ with $ab = 0$, then $ba = 0$.

(B) Suppose D is an integral domain and $a, b, c \in R$ with $a \neq 0$. If $ab = ac$, then $b = c$.

(C) Let n be an integer ≥ 2. Every non-zero element of $\mathbb{Z}/n\mathbb{Z}$ is either a zero-divisor or has a multiplicative inverse.

(D) None of the above.

(v) Select the **True** statement.

(A) Suppose R is an integral domain and I is an ideal of R. If $I \neq R$, then R/I is an integral domain.

(B) If R is a commutative ring and I is an ideal of R, then the characteristic of R equals the characteristic of R/I.

(C) Suppose R is a commutative ring and I is an ideal of R. If R/I is finite, then R is finite.

(D) None of the above.

Question 2. Let $U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

(i) Determine all matrices A in $\mathrm{GL}(2, \mathbb{R})$ such that $AU = UA$. **[3]**

(ii) Determine all matrices B in $\mathrm{GL}(2, \mathbb{R})$ such that $BL = LB$. **[3]**

(iii) Hence determine $Z(\mathrm{GL}(2, \mathbb{R}))$. **[2]**

Question 3. Let M, N be normal subgroups of a group G. Suppose that $M \cap N = \{e\}$. Prove that $mn = nm$ for all $m \in M$, $n \in N$. **[5]**

Question 4.

(i) Let I, J be ideals of a ring R. Prove that

$$I + J = \{x + y : \ x \in I, \ y \in J\}$$

is an ideal of R. **[4]**

(ii) Let R, S be rings and $\phi : R \to S$ be a homomorphism of rings. Prove that kernel(ϕ) is an ideal of R. **[4]**

Question 5.

(i) Let D be a finite integral domain. Prove that D has a multiplicative identity. Then prove that D is a field. **[5]**

(ii) Give an example of an infinite integral domain which is not a field. **[2]**

Question 6. Recall that an ideal M in a ring R is said to be a *maximal ideal* of R if $M \neq R$ and whenever I is an ideal of R such that $M \subseteq I \subseteq R$, then either $I = M$ or $I = R$. For the ring of integers $\mathbb{Z}$, prove the following.

(i) If p is a prime, then $p\mathbb{Z}$ is a maximal ideal; **[4]**

(ii) If n is not a prime, then $n\mathbb{Z}$ is not a maximal ideal. **[3]**

(You can use the fact that every ideal of $\mathbb{Z}$ is a principal ideal.)

ALL THE BEST

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Questions.

Question 1.

(i) True or False: Let $\pi = (1\ 3\ 4\ 2), \rho = (1\ 3), \sigma = (1\ 2)(3\ 4)$. Then $\pi \circ \rho \circ \sigma$ is an even permutation. **[1]**

(ii) Fill in the blank: Let $\pi = (1\ 3\ 4\ 2), \rho = (1\ 3), \sigma = (1\ 2)(3\ 4)$. Then the order of $\pi \circ \rho \circ \sigma$ is ______. **[1]**

(iii) Fill in the blank: In S_{15}, the number of conjugates of $(1\ 2\ 3\ 4\ 5\ 6\ 7)$ is ______. **[2]**

(iv) Fill in the blank: The multiplicative inverse of $4x + 1$ in $\mathbb{Z}/8\mathbb{Z}[x]$ is ______. **[2]**

(v) True or False: The polynomial $x^3 - 2$ is irreducible over $\mathbb{Z}/19\mathbb{Z}$. **[2]**

(vi) Let $f(x) = x^3 + 5x \in \mathbb{Z}/6\mathbb{Z}[x]$. The *number* of roots of $f(x)$ in $\mathbb{Z}/6\mathbb{Z}$ is **[2]**

(A) 3 (B) 1 (C) 6 (D) None of the other options.

Question 2. Determine all group homomorphisms from S_3 to $(\mathbb{Q}, +)$. **[6]**

Question 3. Let p, q, r be primes with $p < q < r$. Prove that there is no simple group of order pqr. **[7]**

Question 4. Let

$$f(x) = 2x^4 + 2 \ \text{ and } \ g(x) = x^5 + 2 \ \text{ in } \ \mathbb{Z}/3\mathbb{Z}[x].$$

Find a greatest common divisor $d(x)$ of $f(x)$ and $g(x)$ in $\mathbb{Z}/3\mathbb{Z}[x]$. Also find $a(x), b(x) \in \mathbb{Z}/3\mathbb{Z}[x]$ such that

$$f(x)a(x) + g(x)b(x) = d(x).$$

(Simply writing $d(x), a(x), b(x)$ and verifying the above equation will lead to a score of zero if you do not show your work finding these polynomials.) **[9]**

Question 5. Let $\mathbb{F}$ be a field. Let I denote the ideal of all polynomials $f(x)$ in $\mathbb{F}[x]$ such that the sum of the coefficients of $f(x)$ is zero. Show that I is a principal ideal by finding a polynomial $g(x) \in \mathbb{F}[x]$ such that I is the principal ideal generated by $g(x)$. Justify your answer. **[6]**

Question 6. Prove that $f(x) = x^4 + 4x + 1$ is irreducible over $\mathbb{Q}$.
(Hint. After a suitable 'transformation', Eisenstein's criterion can be applied.) **[7]**

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