Birla Institute of Technology & Science, Pilani

First Semester 2021-2022, MATH F241 (Mathematical Methods)

Compre Examination (Part-A, CB)

Time: 90 Min. **Date: May 12, 2022 (Thursday)** **Max. Marks: 45**

1. Answer all parts of a question in continuation.
2. Write END in the answer sheet just after the final solution.

1. (a) Solve the integral equation **[7]**

$$\int_0^\infty F(x)\cos(sx)dx = \begin{cases} 1-s, & \text{when} \quad 0 \le s \le 1 \\ 0, & \text{when} \quad s > 1, \end{cases}$$

with the help of Fourier Cosine/Sine transform. Hence, letting $s \to 0$, find the value of $\int_0^\infty \frac{\sin^2 t}{t^2} dt$.

(b) Solve the IVP: **[7]**

$$u_{tt} = 16u_{xx}, \quad -\infty < x < \infty, t > 0$$

with ICs $u(x,0) = x\sin x$, $u_t(x,0) = 0$, $-\infty < x < \infty$ and $u(x,t), u_x(x,t) \to 0$ as $|x| \to \infty$.

(c) Prove that $\mathcal{F}(f(x) * g(x)) = \mathcal{F}(\xi)\mathcal{G}(\xi)$, where $\mathcal{F}(\xi)$ and $\mathcal{G}(\xi)$ are the Fourier transform of $f(x)$ and $g(x)$, respectively. Hence or otherwise, show that **[7]**

$$\frac{d}{dx}\{f(x) * g(x)\} = f^{'}(x) * g(x).$$

2. (a) Find the general solution of the extremal of the functional **[8]**

$$I[z(x,y)] = \int\int \left(\frac{p^2 xy}{2} - \frac{q x^2 y^2}{2} \right) dx\, dy, \quad \text{where} \quad p = \partial z/\partial x, q = \partial z/\partial y.$$

(b) Determine the extremal(s) of the functional **[8]**

$$I[y(x), z(x)] = \int_0^\pi \{2yz - 2y^2 + (y^{'})^2 - (z^{'})^2\} dx$$

which satisfies the boundary conditions $y(0) = z(0) = 0$ and $y(\pi) = z(\pi) = 1$.

(c) Find the extremal of the functional **[8]**

$$\int_0^\pi \{(y^{'})^2 - y^2\} dx$$

under the conditions $y(0) = 0, y(\pi) = 1$ and subject to the constraint $\int_0^\pi y\, dx = 1$.

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Compre Examination (Part-B, OB)

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1. Answer all parts of a question in continuation.
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1. (a) Prove or disprove that **[5]**

$$\int_{-\infty}^{\infty} \delta(x-y)\delta(y-a)dy = \delta(x-a).$$

(b) Find the adjoint equation and adjoint boundary condition of the system **[7]**

$$\left(\frac{x+1}{x}\right)\frac{d^2y}{dx^2} - \left(\frac{1}{x^2}\right)\frac{dy}{dx} + \frac{1}{x^3}y = 0, \quad y(1) + y^{'}(1) = 0, y(2) = 0.$$

Is the operator self-adjoint?

(c) Let $R(x,t;\lambda) = \sum_{m=1}^{\infty} \lambda^{m-1} K_m(x,t)$ be the resolvent kernel of a Fredholm integral equation **[7]**

$$y(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt,$$

then prove or disprove that

$$R(x,t;\lambda) = K(x,t) + \lambda \int_a^b K(x,z)R(z,t;\lambda)dz.$$

(d) Approximate the kernel by the sum of the first two terms in its Taylor series expansion and then solve the integral equation

$$y(x) = e^x - x - \int_0^1 x(e^{xt} - 1)y(t)dt.$$

2. (a) Form the system of equations which determines the approximate values of the solution of equation

$$x^2 = \int_0^1 K(x,\xi)y(\xi)\, d\xi$$

at the points $x_1 = 0, x_2 = 1/4, x_3 = 1/2, x_4 = 3/4$ and $x_5 = 1$ with kernel

$$K(x,\xi) = \begin{cases} x(1-\xi), & \text{when} \quad x > \xi \\ \xi(1-x), & \text{when} \quad x < \xi. \end{cases}$$

(b) Solve the functional

$$I[y] = \int_0^1 \{(y^{'})^2 + 2xy\}dx$$

by using Rayleigh-Ritz method. Use cubic polynomial trial function $y(x) \approx c_0 + c_1x + c_2x^2 + c_3x^3$ with $y(0) = 0$ and $y^{'}(1) = 1$.