BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE, PILANI
K K BIRLA – GOA CAMPUS

SECOND SEMESTER 2022-2023

COMPREHENSIVE EXAMINATION (CLOSED BOOK)

MATHEMATICAL METHODS **MATH F241**

Date: May 11, 2023 Time: 3 Hours

Day: Thursday Max. Marks: 80

INSTRUCTIONS: 1. There are 6 questions. All questions are compulsory. 2. Write all the steps clearly and give explanations for full credit. 3. Number all the pages of your answer book and **make a question-page index** on the front page of the main answer sheet. A penalty of **two marks** will be imposed in case the index is incomplete. 4. Calculator exchange is not allowed.

1. **(a)** Let $f(x)$ be a continuous function in $[a,b] \subset \mathbb{R}$ such that $\int_a^b f(x)g'(x)dx = 0$ for every function $g(x) \in C^1(a,b)$ with $g(a) = 0 = g(b)$. Show that $f(x)$ is constant for all $x \in [a,b]$. **[8]**

   (b) Find the extremal of the isoparametric problem $J(y(x)) = \frac{1}{2}\int_0^{\pi/2} \left(y^2 - y'^2\right) dx$ with boundary conditions $y(0) = 0,\ y(\pi/2) = 1$ and the subsidiary condition $\int_0^{\pi/2} y\,dx = 3 - \frac{\pi}{2}$. **[7]**

2. **(a)** If $y(x)$ is an extremizing function for **[8]**

$$J(y) = \int_{x_0}^{x_1} F(x,y,y')\,dx, \quad y(x_0) = y_0 \text{ and } y(x_1) = y_1.$$

   Then show show that the first variation $\delta J(y) = 0$.

   (b) Find an approximate solution of the following problem of minimizing the functional using Ritz method (Use one term approximation to write your solution) **[7]**

$$J(y(x)) = \int_0^1 \left(y'^2 + y^2 + 2xy\right) dx,\ y(0) = 0 \text{ and } y(1) = 1.$$

3. Find the extremal of the following functional **[10]**

$$J(u(x,y)) = \iint_R \left(u_x^2 + u_y^2 + 2uf(x,y)\right) dxdy.$$

   If $f(x,y) = 0$ and $u(x,y)$ satisfies the initial condition $u(x,0) = 2$ with $u(x,y) \to 0$ as $y \to \infty$, $-\infty < x < \infty$, then find the solution of the above extremal using Fourier Transform.

4. **(a)** Construct the Green's function for the boundary value problem (BVP) **[8]**

$$y'' + 4y = 0, \quad y(0) = 0, \quad y(1) = 0.$$

(b) Find the resolvent kernel and solve the following integral equation **[7]**

$$y(x) = x + \int_0^x (t - x)y(t)dt.$$

5. **(a)** Solve the following Fredholm integral equation **[7]**

$$y(x) = x + 2\int_0^1 (xt^2 + x^2t)y(t)dt.$$

(b) Transform the following BVP **[10]**

$$\frac{d^2y}{dx^2} + xy = 1, \quad y(0) = y(1) = 0$$

into integral equation. Conversely, show that the reduced integral equation satisfies the BVP.

6. Let $f(x)$ is a piece wise continuously differentiable and absolutely integrable function with $F(\omega) = \mathcal{F}[f(x)]$. Then show that **[8]**

(i) $F(\omega)$ is bounded for $-\infty < \omega < \infty$,

(ii) $F(\omega)$ is continuous for $-\infty < \omega < \infty$.

$* * * * * *$ **ALL THE BEST** $* * * * * *$