BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE, PILANI
K K BIRLA – GOA CAMPUS

SECOND SEMESTER 2022-2023

MID SEMESTER EXAM (CLOSED BOOK)

MATHEMATICAL METHODS | **MATH F241**
Date: March 17, 2023 | Time: 90 Minutes
Day: Friday | Max. Marks: 60

INSTRUCTIONS: 1. There are 5 questions. All questions are compulsory. 2. Write all the steps clearly and give explanations for full credit. 3. Number all the pages of your answer book and **make a question-page index** on the front page of the main answer sheet. A penalty of **two marks** will be imposed in case the index is incomplete. 4. Calculator exchange is not allowed.

1. **(a)** Find the Fourier series of the function $f(x) = |x|, -\pi < x < \pi$ such that $f(x+2\pi) = f(x)$ and use it to prove that **[8]**

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

(b) If $\tilde{y}(x)$ is an extremal of the functional $J(y) = \int_{x_1}^{x_2} F(x, y, y')dx$, then show that the Euler-Lagrange equation can also be written as **[7]**

$$\frac{d}{dx}\left(F - \tilde{y}'\,\frac{\partial F}{\partial \tilde{y}'}\right) - \frac{\partial F}{\partial x} = 0.$$

Also discuss the case when F is independent of x.

2. **(a)** Show that the Euler-Lagrange equation for the functional $J(y) = \int_{x_1}^{x_2} f(x,y)\sqrt{1+y'^2}dx$ has the form **[9]**

$$f_y - f_x\, y' - \frac{f\, y''}{1+y'^2} = 0.$$

(b) Use it to find an extremal for the functional

$$J(y) = \int_1^2 \frac{\sqrt{1+y'^2}}{x}dx, \quad y(1) = y(2) = 1.$$

[6]

 P.T.O.

3. **(a)** Prove the following integral identity **[5]**

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} F(\omega)\overline{G(\omega)}d\omega,$$

where $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(x)$ and $g(x)$, respectively.

(b) Use the appropriate Fourier transform (sine/cosine) to evaluate the integral **[5]**

$$\int_{0}^{\infty} \frac{dx}{(x^2+9)^2}$$

4. **(a)** Prove the following Parseva'l theorem for Fourier series on $[-\pi, \pi]$ **[5]**

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2 dx,$$

where a_0, a_n and b_n are the Fourier coefficients of $f(x), -\pi < x < \pi$.

(b) If $\mathcal{F}(f(x)) = F(\omega)$ and $\mathcal{F}(g(x)) = G(\omega)$ then show that **[5]**

$$\mathcal{F}[(f * g)(x)] = F(\omega)G(\omega),$$

where $(f * g)(x)$ represents the convolution of $f(x)$ and $g(x)$.

5. **(a)** Solve the following ODE using Fourier transform **[5]**

$$3\frac{dy}{dt} + 2y = \delta(t),$$

where $y(t) \to 0$ as $t \to \infty$.

(b) Solve the following PDE using Fourier transform **[5]**

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \delta(x)\delta(t), -\infty < x < \infty, t > 0,$$

$$u(x,0) = \delta(x),\ u(x,t) \to 0 \text{ as } |x| \to \infty.$$

$* * * * * *$ **ALL THE BEST** $* * * * * *$