

Birla Institute of Technology and Science, Pilani 333031
II Semester 2021-22
MATH F244 (Measure & Integration)
Comprehensive Exams Part-B (Open Book)

Max. Marks 40

Date: 10/05/2022

Time: 90 Min.

Q.1. Find the limit and justify your steps

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{(nx)^2}{(1+x^2)^n} dx \quad [8]$$

Q.2. Let $E \subset \mathbb{R}$ be a measurable subset with $m(E) = 1$ and let $f \in \mathcal{L}^1(E)$ satisfies $f(x) \geq M > 0$ a. e. on E ,

(a) Show that $h(x) = \ln f(x)$, is a measurable function for $f(x) > 0$.

(b) Show that $g(t) = t - 1 - \ln t$ has minima at $t = 1$.

(c) Use (b) to show $1 - 1/t \leq \ln t \leq t - 1$.

(d) Use (c) to show that $h \in \mathcal{L}^1(E)$ and $\|h\|_1 \leq \ln(\|f\|_1)$. [2 + 2 + 3 + 3 + 5]

Q.3. Prove or disprove the following:

(a) There exist a non-measurable subset $A \subset \mathbb{R}$ such that the set $B = \{x \in A: x \text{ is irrational}\}$ is measurable.

(b) There exist a non-measurable function $f \geq 0$, s.t. \sqrt{f} is a measurable function.

[4 + 4]

Q.4. Compute total variation $\{T_{-2}^2(f)\}$, positive variation $\{P_{-2}^2(f)\}$ and negative variation $\{N_{-2}^2(f)\}$ for the function $f(x) = 6x^2 - 4x^3$. [9]