Birla Institute of Technology & Science, Pilani

Mid Semester Examination (Closed Book), Second Semester 2022-23

Measure & Integration (MATH F244)

Date: March 18, 2023 **Max. Time: 90 Minutes** **Max. Marks: 70**

Note. Question no. 1 is compulsory. Attempt any five questions from Question no. 2 to 8. Start answering each question on a new page.

1. Define the following terms:
   (i) Outer measure of a set.
   (ii) Measurable set.
   (iii) Borel sets in $\mathbb{R}$.
   (iv) Measurable function.
   (v) Upper Lebesgue integral and lower Lebesgue integral of a bounded function. [2×5 **Marks**]

2. Let $a < b$ be real numbers. Then prove that $m^*([a, b]) = b - a$. **[12 Marks]**

3. Let $E_1, E_2, \ldots, E_n$ be disjoint measurable sets. Then prove that

$$m(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m(E_i)$$

[12 Marks]

4. Prove that there exists a non-measurable set in the interval $[0, 1]$. **[12 Marks]**

5. Let E be a measurable set with $m(E) < \infty$, and let $\{f_n\}$ be a sequence of measurable functions defined on E. Let f be a measurable function such that $f_n(x) \to f(x)$ for all $x \in E$. Then prove that for given $\epsilon > 0$ and $\delta > 0$, there is a measurable set $A \subset E$ with $m(A) < \delta$ and an integer N such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $x \in E - A$ and all $n \geq N$. **[12 Marks]**

6. State and prove the Fréchet theorem. **[3+9 Marks]**

7. Let $a < b$ be real numbers. Then prove that every bounded Riemann integrable function over a closed interval $[a, b]$ is Lebesgue integrable. Is the converse true? **[8+4 Marks]**

8. State and prove the bounded convergence theorem. **[3+9 Marks]**

—— Good Luck ——