Birla Institute of Technology & Science, Pilani

Comprehensive Examination (Closed Book), Second Semester 2022-23

Measure & Integration (MATH F244)

Date: May 20, 2023 **Max. Time: 155 Minutes** **Max. Marks: 81**

Note. Start answering each question on a new page.

1. Do any **one** of the following:

   (i) Let $\{E_i\}$ be an infinite decreasing sequence of measurable sets, that is, a sequence with $E_{i+1} \subset E_i$ for each $i \in \mathbb{N}$. Let $m(E_i) < \infty$ for at least one $i \in \mathbb{N}$. Then show that

   $$m(\bigcap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} m(E_n)$$

   (ii) Let $\{E_i\}$ be an infinite increasing sequence of sets. Then show that

   $$m(\bigcup_{i=1}^{\infty} E_i) = \lim_{n\to\infty} m(E_n)$$

   [9 Marks]

2. Let f be a bounded measurable function defined on a measurable set E with $m(E) < \infty$. Show that if $f = 0$ a.e on E, then $\int_E f = 0$. Also, show that the converse is true when $f(x) \leq 0$ on E. **[9 Marks]**

3. Do any **two** of the following:

   (i) State and prove the Fatou's lemma and the monotone convergence theorem.

   (ii) Let E be a measurable set and let f, g be two integrable functions on E. Then show that $f + g$ is integrable on E. Moreover, prove that $\int_E (f + g) = \int_E f + \int_E g$

   (iii) Let $-1 < a < 0$ and the function $f : [0, 1] \to \mathbb{R}$ be defined by

   $$f(x) = \begin{cases} x^a & \text{for } 0 < x \leq 1 \\ 0 & \text{for } x = 0 \end{cases}$$

   Calculate $\int_0^1 f$ **[9+9 Marks]**

4. Do any **three** of the following:

   (i) Let f be a bounded and measurable function defined on $[a, b]$. If

   $$F(x) = \int_a^x f(t)\ dt + F(a)$$

   then show that $F'(x) = f(x)$ a.e. in $[a, b]$.

   (ii) Let f be a function of bounded variation on $[a, b]$. Then show that f is continuous at a point in $[a, b]$ if and only if its variation function v_f is continuous at that point.

   (iii) Let f be an integrable function on $[a, b]$. Define Lebesgue point of f. Let x be a Lebesgue point of f. Then show that the indefinite integral

   $$F(x) = \int_a^x f(t)\ dt + F(a)$$

   is differentiable at x and $F'(x) = f(x)$. Moreover, show that every point of continuity of f is a Lebesgue point of f.

(iv) If F is an absolutely continuous function on $[a, b]$, then show that F is an indefinite integrable of its derivative. **[10+10+10 Marks]**

5. State Vitali's covering theorem. Show that the union of any collection of closed intervals is a measurable set. **[2+4 Marks]**

6. State and prove the Riesz-Minkowski inequality for $1 \leq p \leq \infty$. **[9 Marks]**

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Comprehensive Examination (Open Book), Second Semester 2022-23

Measure & Integration (MATH F244)

Date: May 20, 2023 **Max. Time: 25 Minutes** **Max. Marks: 9**

1. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x+1 & \text{for } x < -3 \\ 0 & \text{for } -3 \leq x < -2 \\ -|x| & \text{for } -2 \leq x < 2 \\ 1 & \text{for } 2 \leq x < 3 \\ -x^2 & \text{for } 3 \leq x < 4 \\ x^3 & \text{for } x \geq 4 \end{cases}$$

   Find the set $\{x \in \mathbb{R} : f(x) \leq \alpha\}$ for all $\alpha \in \mathbb{R}$. Is f measurable? **[9 Marks]**

—— Good Luck ——