Birla Institute of Technology and Science, Pilani
I-Semester 2016-17
(Introduction to Topology) MATH F311
Mid-Semester Exams (Close Book)

Max. Marks 60 Date: 6th October 2016 Time: 90 Min.

Q.1 Let τ be the collection of subsets of $\mathbb{N}$, which contains $\mathbb{N}$, φ and all *finite* subsets of $\mathbb{N}$. Is τ a topology on $\mathbb{N}$? Justify. [6]

Q.2 Define *basis* for a topology on X. If $\mathbb{B}$ is a basis on X, then how would you define a topology on X? Justify. [3+3+6]

Q.3 Let $\tau = \{G \subset \mathbb{R}: x \in G \Leftrightarrow -x \in G\}$ be a topology on $\mathbb{R}$. Then:

(a) Show that $\mathbb{Z}$ and $\mathbb{Q}$ are τ-*clopen* subsets but $\mathbb{N}$ is neither *open* nor *closed*.

(b) Find *closure* of $\mathbb{N}$. Justify. [9+5]

Q.4 Suppose X, Y are topological spaces, and f: $X \rightarrow Y$ is a function.

(a) Define *continuity* of $f(x)$ at a point $x_0 \in X$.

(b) Let f is *continuous*. In the space $X \times Y$ (with the product topology) we define a *subspace* G called the "*graph of f*" as $G = \{(x, y) \in X \times Y \mid y = f(x)\}$. Prove that G is *homeomorphic* to X. (State clearly whatever theorem(s) you use in doing this proof.) [3+6+4]

Q.5 Let $X = \mathbb{R}^\omega$ with the *box topology*. Let $A \subset \mathbb{R}^\omega$ consist of the points $(x_1, x_2 \ldots)$ with all $x_i > 0$.

(a) Show that $\mathbf{0} = (0, 0 \ldots) \in \overline{A}$.

(b) Show that a sequence of points in A cannot *converge* to $\mathbf{0}$.

(c) What does the *sequence lemma* imply about the *metrizability* of X? [5+5+5]

Solutions

Ans.1 No

Let $\{U_n\}$ where $U_n = \{n\}$, $\forall n = \{2, 3, 4\ldots\}$ be the countable collection of τ-open subsets of ℕ, but $\bigcup_{n=2}^{\infty} U_n = \{2,3,4\ldots\} \notin \tau$.

Ans.2 Basis: Let X be a non-empty set, a collection $\mathbb{B}$ of subsets of X (called basis element) such that:

[B1] $\forall x \in X$, there is at least one basis element $B \in \mathbb{B}$ containing x

[B2] If $x \in B_1 \cap B_2$ then $\exists$ $B_3 \in \mathbb{B}$ containing x s.t. $x \in B_3 \subset B_1 \cap B_2$.

We define a topology τ on X by defining a subset G of X to be τ-open in X, if $\forall x \in G$, there is a basis element $B \in \mathbb{B}$ s.t. $x \in B \subset G$.

Now we show that τ is a topology on X.

[T_1] $\varphi \in \tau$ (vacuously!), $X \in \tau$ by [B1]

[T_2] Let $\{U_\alpha\}_{\alpha \in J}$ be an indexed family of τ-open subsets of X and $U = \bigcup_{\alpha \in J} U_\alpha$. Let $x \in U$, there is an index α, such that $x \in U_\alpha$. Since U_α is τ-open, there is a basis element $B \in \mathbb{B}$ s.t. $x \in B \subset U_\alpha$. Then $x \in B \subset U$ and therefore U is τ-open.

[T_3] Let U_1 and U_2 are τ-open subset of X. Let $x \in U_1 \cap U_2$. Then $\exists$ B_1 and B_2 in $\mathbb{B}$ s.t. $x \in B_1 \subset U_1$ and $x \in B_2 \subset U_2$. By [B2], $\exists$ $B_3 \in \mathbb{B}$, s.t. $x \in B_3 \subset B_1 \cap B_2 \subset U_1 \cap U_2$. So $U_1 \cap U_2$ is τ-open.

Ans.3 (a) ℤ is τ-open as $m \in \mathbb{Z} \Leftrightarrow -m \in \mathbb{Z}$

ℤ is τ-closed as $\mathbb{R}\sim\mathbb{Z}$ is open since $m \in \mathbb{R}\sim\mathbb{Z} \Leftrightarrow -m \in \mathbb{R}\sim\mathbb{Z}$.

Similar argument can be given for ℚ.

ℕ is neither τ-open as $1 \in \mathbb{N}$ but $-1 \notin \mathbb{N}$. ℕ is nor τ-closed as $\mathbb{R}\sim\mathbb{N}$ is not τ-open since $-1 \in \mathbb{R}\sim\mathbb{N}$, but $1 \notin \mathbb{R}\sim\mathbb{N}$.

(b) τ-closure of $\mathbb{N} = \mathbb{Z} \sim \{0\}$. Since if $m \in \mathbb{Z}^+$, then $m \in \mathbb{N}$ and if $m \in \mathbb{Z}^-$, every τ-nbd U of m contains $-m \in \mathbb{N}$ so $U \cap \mathbb{N} \neq \varphi$, so m is limit point of $\mathbb{N}$. Other than these numbers no real number is limit point of $\mathbb{N}$.

Ans.4 (a) $f(x)$ is continuous at a point $x_0 \in X$ if for each nbd V of $f(x_0)$, $\exists$ a nbd U of x_0 s.t. $f(U) \subset V$.

(b) To prove this result, we use following theorems:

Theorem 1: The projection map $\pi: X \times Y \to X$ is continuous.

Theorem 2: The restriction of a continuous function to a subspace is continuous. (Theorem 18.2 (d))

Define $\varphi: X \to G$ by $\varphi(x) = (x, f(x))$. As the Cartesian product of two continuous functions is continuous, φ is continuous. Check that φ is a bijection [1-1 is because f is a function; surjective is by definition of the "graph of f"]. The function φ^{-1} is just the restriction to G of the projection map $\pi: X \times Y \to X$. π is continuous and the restriction of a continuous function to a subspace is continuous. So $\varphi^{-1}: G \to X$ is continuous.

Ans.5 (a) We need to show that every basis element that contains **0** contains a point of A other than **0**.

If a basis element $U = (a_1, b_1) \times (a_2, b_2) \times \ldots$ contains **0**, then all $b_i > 0$. The point $(b_1/2, b_2/2 \ldots)$ is contained in $U \cap A$, showing that this set is not empty.

(b) Consider a sequence $\boldsymbol{x}_1 = < x_{11}, x_{12} \ldots >$, $\boldsymbol{x}_2 =< x_{21}, x_{22} \ldots >$, $\boldsymbol{x}_3 =< x_{31}, x_{32} \ldots > \ldots$ of points in A. Then all $x_{ij} > 0$, and the basis element $U = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times \ldots$ is a neighborhood of **0**. But the neighborhood U does not contain any of the points $\boldsymbol{x}_i$ of the sequence, so the sequence does not converge to **0**.

(c) The sequence lemma (Lemma 21.2, page 130) says that in a metrizable topological space each point $x \in \bar{A}$ is the limit of some sequence of points in A. We have here an example where a point in the closure $\bar{A}$ is NOT the limit of any sequence, so we conclude that $X = \mathbb{R}^\omega$ with the box topology is NOT metrizable.