Birla Institute of Technology & Science, Pilani (Raj.)

First Semester 2023-2024, MATH F311 (Introduction to Topology)
Mid-Semester Examination (Closed Book)

Time: 90 Minutes **Date: October 11, 2023 (Wednesday)** **Max. Marks: 50**

1. There are total 5 questions. Start answering each question on a new page. Answer each sub-part of the questions in continuation. While answering, write all the steps with proper justification. Write "END" after the last attempted solution.
2. All notations used in the paper have the usual meaning.

Q 1. State whether the following statements are true or false. Justification is not required. **[10]**

(i) The map $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined as $d(x, y) = |e^x - e^y|$, is a metric on $\mathbb{R}$.

(ii) Let $X = \{a, b, c, d, e\}$ and $\mathscr{T} = \big\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}\big\}$. Then $\mathscr{T}$ is a topology on X.

(iii) $\mathscr{B} = \{(a, b] : a, b \in \mathbb{R}\}$ is a basis for the lower limit topology on $\mathbb{R}$.

(iv) Consider $\mathbb{R}$ with the standard topology and $Y = [0, 1] \cup \{2\}$. Then $[0, 1]$ is open in the subspace topology of Y.

(v) Consider $\mathbb{R}$ with the lower limit topology. Then $[a, b)$ is open.

(vi) Consider $\mathbb{R}$ with the lower limit topology. Then $[a, b)$ is closed.

(vii) Let X be a discrete topological space and Y be any topological space. Then $f : X \to Y$ is continuous.

(viii) Let $\{X_\alpha\}_{\alpha \in I}$ be an uncountable collection of topological spaces. Then the box topology and the product topology on $\prod_{\alpha \in I} X_\alpha$ are same.

(ix) A bijective quotient map is not a homeomorphism in general.

(x) The map $f : \mathbb{R} \to [-1, 1]$, defined as $f(x) = \sin x$, is a homeomorphism.

Q 2. (i) Define standard and cofinite topology on $\mathbb{R}$. **[2+2]**

(ii) Show that the cofinite topology is strictly smaller than the standard topology on $\mathbb{R}$. **[6]**

Q 3. Consider $\mathbb{R}$ with the standard topology and $A = (0, 1) \cap \mathbb{Q}$.

(i) Find the set of all limit points and the closure of A. **[5]**

(ii) Find the set of all boundary points and the interior of A. **[5]**

Q 4. (i) Let X be a topological space and Y be a subset of X. Define the subspace topology on Y. **[2]**

(ii) Consider $\mathbb{R}$ with the standard topology. Then what is the subspace topology on $\mathbb{N}$? **[4]**

(iii) Suppose that X is a topological space with the discrete topology. Is the subspace topology of a subset $Y \subseteq X$ necessarily the discrete topology on Y? Justify. **[4]**

Q 5. (i) Let $X = \{a, b, c, d\}$ and $\mathscr{T}_1, \mathscr{T}_2$ be two topologies on X given by

$$\mathscr{T}_1 = \big\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\big\}, \quad \mathscr{T}_2 = \big\{X, \emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\big\}.$$

Consider $f : (X, \mathscr{T}_1) \longrightarrow (X, \mathscr{T}_2)$ defined as

$$f(a) = b,\ f(b) = c,\ f(c) = d,\ f(d) = a.$$

Is f continuous? Justify. **[3]**

(ii) Define homeomorphism between two topological spaces. **[2]**

(iii) Show that $(0, 1)$ and $(0, \infty)$ are homeomorphic. **[5]**

END