Birla Institute of Technology & Science, Pilani (Raj.)

First Semester 2023-2024, MATH F311 (Introduction to Topology)
Comprehensive Examination (Closed Book)

Time: 180 Minutes **Date: December 11, 2023 (Monday)** **Max. Marks: 90**

1. There are total 6 questions. Start answering each question on a new page. Answer each sub-part of the questions in continuation. While answering, write all the steps with proper justification. Write "END" after the last attempted solution.
2. All notations used in the paper are standard and have their usual meaning.

Q 1. State whether the following statements are true or false. Justification is not required. **[10]**

(i) Consider $\mathbb{R}$ with the standard topology. Then the subspace topology on $Y = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ is discrete.

(ii) A homeomorphism is an open map.

(iii) In every topological space, a convergent sequence has a unique limit.

(iv) $\mathbb{Q}$ with the cofinite topology is connected.

(v) Every subspace of a compact space is compact.

(vi) A subspace of a metric space is compact if and only if it is closed and bounded.

(vii) The one point compactification of $\mathbb{R}$ is homeomorphic to the sphere.

(viii) The product of regular spaces is regular.

(ix) The product of normal spaces is normal.

(x) A subspace of a Lindelöf space is Lindelöf.

Q 2. Answer the following questions. Justification is not required. **[10]**

(i) What is the co-countable topology on $\mathbb{Q}$?

(ii) Consider $\mathbb{R}$ with the lower limit topology. Let $A = (0, 1) \cap \mathbb{Q}$. Then what is $\overline{A}$ and A'?

(iii) Let $\mathbb{R}_\ell$ and $\mathbb{R}_s$ denote the real line with the lower limit topology and the standard topology, respectively. Let $f : \mathbb{R}_s \to \mathbb{R}_\ell$ be defined as $f(x) = x$ for all $x \in \mathbb{R}$. Is f continuous?

(iv) Consider $[0, 1]$ with the discrete topology. What is the limit of the sequence $(1/n)_{n\geq 1}$?

(v) How many components does a topologist's sine curve have? How many path components does a topologist's sine curve have?

(vi) State Tychonoff's theorem.

(vii) Let X be a finite set. If a topology on X makes it Hausdorff, then what is the topology on X?

(viii) State all the countability axioms satisfied by the real line with the lower limit topology.

Please turn over...

Q 3. (i) Show that the collection $\mathscr{B} = \Big\{[a, b) : a < b \text{ and } a, b \in \mathbb{Q}\Big\}$ is a basis. Is the topology generated by $\mathscr{B}$ equal to the lower limit topology on $\mathbb{R}$? Justify. **[10]**

(ii) Consider $\mathbb{R}$ with the standard topology. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at precisely one point. Justify with details. **[8]**

Q 4. (i) Define connected space. **[3]**

(ii) Let X, Y be topological spaces and $p : X \to Y$ be a quotient map. Show that if each set $p^{-1}(\{y\})$ is connected, and if Y is connected, then X is connected. **[14]**

Q 5. (i) Define compact space and define Hausdorff space. **[3+3]**

(ii) Prove that if Y is a compact subspace of the Hausdorff space X and $x_0 \notin Y$, then there exist disjoint open sets U and V containing x_0 and Y, respectively. **[12]**

Q 6. (i) Define dense subset, separable space and normal space. **[2+3+3]**

(ii) Show that an open subspace of a separable space is separable. **[3]**

(iii) Show that a closed subspace of a normal space is normal. Give an example to show that a subspace of a normal space need not be normal. Justify with details. **[6]**

—————— **END** ——————