BIRLA INSTITUTE OF TECHNOLOGY AND SCIENCE, PILANI
First Semester 2016 – 17
MATH F312 : Ordinary Differential Equations
Comprehensive Examination

Duration : 3 hrs   Date : 14/12/2016   MM : 90

Part-A: Closed book   **Max Marks: 20**   **Max Time: 40 mins**   SET A

Name: **ID No.:**

Note: Write only the correct options (A, B, C or D in CAPITAL LETTER ONLY) of the given questions in the box provided below. Each correct answer carries 2 marks and incorrect answer carries -0.5 mark. Over writing/cutting will be treated as unattempted.

Q. No.	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Answer										

A1. Let $u_1(t)$ and $u_2(t)$ are the solutions of $u'' + ku' + w^2u = 0$, k & w are constants. Then

$v = u_1^2\left(\frac{u_2}{u_1}\right)'$ is a solution of the differential equation

(A) $v' - kv = 0$
(B) $v' + w^2v = 0$
(C) $v' - w^2v = 0$
(D) None of these

A2. All nonnegative continuous function $u(t)$ on $0 \leq t \leq 1$ satisfying

$u(t) \leq \int_0^t u(s)ds$, $0 \leq t \leq 1$ are given by

(A) $u(t) = e^t$
(B) $u(t) = e^{-t}$
(C) $u(t) = 0$
(D) None of these

A3. For the differential equation $u'' + \frac{6}{t}u' + \frac{6}{t^2}u = 0$, which of the following statements is true?

(A) $u_1(t) = \frac{1}{t^2}$, $u_2(t) = \frac{1}{t^3}$ form a fundamental system of solutions
(B) $u_1(t) = \frac{1}{t^2}$, $u_2(t) = \frac{1}{t^3}$ can not form a fundamental system of solutions
(C) $u_1(t) = t^2$, $u_2(t) = t^3$ form a fundamental system of solutions
(D) None of these

A4. Let $u(t)$ be any nontrivial solution of $u''' + 2u'' + 3u' + 4u = 0$. Then which of the following statements is true?

(A) $\lim_{t\to\infty} u(t) = 0$
(B) $\lim_{t\to\infty} u(t) = \infty$
(C) $\lim_{t\to\infty} u(t) = c$, $c \neq 0$
(D) None of these

A5. Let $u(t)$ be any nontrivial solution of the differential equation $u'' + e^t\, u = 0$ on $[0, \infty)$. Then which of the following statements is true?

(A) $u(t)$ is unbounded

(B) $u(t)$ is bounded

(C) $u(t)$ is bounded in $[0, 1)$ but unbounded on $[1, \infty)$

(D) none of these

A6. The real equilibrium point(s) of the system

$$x_1' = x_1^3 + x_2,\ x_2' = x_1 + x_2^3 \quad \text{is(are)}$$

(A) (0, 0) only

(B) $(0,0), (1,1)$ and $(-1,-1)$

(C) $(0,0), (1,-1)$ and $(-1,1)$

(D) not having any real equilibrium point

A7. Let

$$\Phi(t) = \begin{pmatrix} 1 & t^2 & t^4 \\ 0 & e^t & e^{-t} \\ 0 & 0 & 0 \end{pmatrix},\ \phi_1(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \phi_2(t) = \begin{pmatrix} t^2 \\ e^t \\ 0 \end{pmatrix},\ \ \phi_3(t) = \begin{pmatrix} t^4 \\ e^{-t} \\ 0 \end{pmatrix},\ a < t < b.$$

Then which of the following statements is true?

(A) $\phi_1(t),\ \phi_2(t),\ \phi_3(t)$ are linearly independent on (a, b)

(B) $\phi_1(t),\ \phi_2(t),\ \phi_3(t)$ are linearly dependent on (a, b)

(C) Linearly dependence/independence can not be determined as $\det \Phi(t) = 0$

(D) None of these

A8. The zero solution of the IVP: $u' + 2u + e^t u^2 = 0, \quad u(0) = u_0 > 0$

(A) stable but not asymptotically stable

(B) asymptotically stable

(C) unstable

(D) None of these

A9. Let $y_1(t), y_2(t), \dots\ , y_n(t)$ be any solutions of the differential equation

$$y^{(n)} + a_2(t)y^{(n-2)} + \ a_3(t)y^{(n-3)} + \cdots + a_{n-1}(t)y' + a_n(t)y = 0,$$

and $W(t)$ be the Wronskian of $y_1(t), y_2(t), \dots\ , y_n(t)$ with $W(t_0) = 1$ Then which of the following statement is true?

(A) $W(t)$ is an increasing function of t

(B) $W(t)$ is a decreasing function of t

(C) $W(t)$ is always a constant

(D) None of these

A10. The equilibrium point (0, 0) of the system of differential equations

$$x_1' = -x_1^3 - 4x_2, \qquad x_2' = 3x_1 - x_2^3 \qquad \text{is}$$

(A) a saddle point

(B) unstable spiral

(C) asymptotically stable

(D) None of these

*****END OF PART A*****

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Duration : 3 hrs   Date : 14/12/2016   MM : 90

Part-A: Closed book   **Max Marks: 20**   **Max Time: 40 mins**   SET B

Name: **ID No.:**

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Q. No.	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Answer										

A1. Let $y_1(t), y_2(t), \dots ,y_n(t)$ be any solutions of the differential equation

$$y^{(n)} + a_2(t)y^{(n-2)} + a_3(t)y^{(n-3)} + \cdots + a_{n-1}(t)y' + a_n(t)y = 0,$$

and $W(t)$ be the Wronskian of $y_1(t), y_2(t), \dots ,y_n(t)$ with $W(t_0) = 1$ Then which of the following statement is true?

(A) $W(t)$ is an increasing function of t
(B) $W(t)$ is a decreasing function of t
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(D) None of these

A2. Let $u_1(t)$ and $u_2(t)$ are the solutions of $u'' + ku' + w^2u = 0$, k & w are constants. Then

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(D) None of these

A5. Let $u(t)$ be any nontrivial solution of the differential equation $u'' + e^t u = 0$ on $[0, \infty)$. Then which of the following statements is true?

(A) $u(t)$ is unbounded
(B) $u(t)$ is bounded
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A6. The equilibrium point (0, 0) of the system of differential equations

$$x_1' = -x_1^3 - 4x_2, \quad x_2' = 3x_1 - x_2^3 \quad \text{is}$$

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(B) unstable spiral
(C) asymptotically stable
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A7. Let $u(t)$ be any nontrivial solution of $u''' + 2u'' + 3u' + 4u = 0$. Then which of the following statements is true?

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$$x_1' = x_1^3 + x_2, \; x_2' = x_1 + x_2^3 \quad \text{is(are)}$$

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(D) not having any real equilibrium point

A10. Let

$$\Phi(t) = \begin{pmatrix} 1 & t^2 & t^4 \\ 0 & e^t & e^{-t} \\ 0 & 0 & 0 \end{pmatrix}, \; \phi_1(t) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; \phi_2(t) = \begin{pmatrix} t^2 \\ e^t \\ 0 \end{pmatrix}, \; \phi_3(t) = \begin{pmatrix} t^4 \\ e^{-t} \\ 0 \end{pmatrix}, \; a < t < b.$$

Then which of the following statements is true?

(A) $\phi_1(t),\ \phi_2(t),\ \phi_3(t)$ are linearly independent on (a, b)
(B) $\phi_1(t),\ \phi_2(t),\ \phi_3(t)$ are linearly dependent on (a, b)
(C) Linearly dependence/independence can not be determined as $\det \Phi(t) = 0$
(D) None of these

*****END OF PART A*****

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Duration : 3 hrs | Date : 14/12/2016 | MM : 90

Part-B: Closed book | **Max Marks: 30** | **Max Time: 60 mins**

Note: 1. Start answer of each question on a fresh page.
2. Write Part B on the left right top corner of the answer book.
3. Write END in the last when you have finished your answer, and strike off the blank pages and rough work.

B1. Let $a(t)$ be a continuously differentiable function for $t \in [0, \infty)$. If $a(t) \to \infty$ monotonically as $t \to \infty$, then show that all solutions of $u'' + a(t)u = 0$ are bounded on $[0, \infty)$. [10]

B2. Let $g(t)$ and $h(t)$ be continuous functions on the interval $J = [t_0, t_0 + a]$, where a is fixed real number. Then show that the IVP:

$$u'' + g(t)u' + h(t)u = 0,\ u(t_0) = u_0,\ u'(t_0) = u_1$$

has a unique solution on J. [10]

B3.Let $f(t)$ be continuous on $0 \leq t < \infty$ and $f(t) \to c\ as\ t \to \infty$ with $c > 0$. Using the Sturm Comparison Theorem, show that all nontrivial solutions of $u'' + f(t)u = 0$ are oscillatory on $0 \leq t < \infty$. [10]

END OF PART B

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Duration : 3 hrs | Date : 14/12/2016 | MM : 90

Part-C: Open Book | **Max Marks: 40** | **Max Time: 80 mins**

Note: 1. Start answer of each question on a fresh page.
2. Write Part B on the left right top corner of the answer book.
3. Write END in the last when completed all answers and strike off the blank pages & rough work.

C1. Find all possible limit cycles and discuss it stability behavior for the following system of differential equations:

$$x_1' = x_2 - x_1(x_1^2 + x_2^2 - 36)^2,$$

$$x_2' = -x_1 - x_2(x_1^2 + x_2^2 - 36)^2.$$ [8]

C2. Consider the following resource-consumer 3D system:

$$x_1' = 5x_1 - x_1^2 - x_1x_2,$$

$$x_2' = -2x_2 + 2x_1x_2 - x_2x_3,$$

$$x_3' = -6x_3 + 3x_2x_3.$$

For the above system, answer the following questions:

(i) Find a positive equilibrium (x_1^*, x_2^*, x_3^*). [2]

(ii) Linearize the given system of equations around the positive equilibrium (x_1^*, x_2^*, x_3^*) obtained in the previous step (i). [4]

(iii) Test the stability behavior of the positive equilibrium (x_1^*, x_2^*, x_3^*) for the linear model obtained in step (ii) using the Routh-Hurwitz criteria. [6]

C3. For the following system of differential equations

$$x_1' = 12x_1\left(1 - \frac{x_1}{2}\right) - 6x_1x_2\ ,$$

$$x_2' = 9x_2\left(1 - \frac{2x_2}{3}\right) - 3x_1x_2\ ,$$

answer the following questions:

(i) Find all possible equilibrium points of the given system of equations. [4]

(ii) Test the stability behavior of each of the equilibrium points by computing the Jacobian matrix corresponding to each equilibrium point. [8]

(iii) Let (x_1^*, x_2^*) be the positive (non-zero) equilibrium point of the given system and consider the positive definite function around the positive equilibrium: $V(x_1, x_2) = (x_1 - x_1^* - x_1^* \ln x_1) + \alpha(x_2 - x_2^* - x_2^* \ln x_2)$, where α is any positive real constant. Write $\frac{dV}{dt}$ in terms of a quadratic equation, then using the Sylvester criteria find a condition on α which makes $\frac{dV}{dt}$ to be negative definite in the interior of the positive quadrant of the x_1x_2 –plane, and hence prove that (x_1^*, x_2^*) is non-linearly asymptotically stable if α satisfies a condition obtained in the previous step. [8]

END OF PART C