**Birla Institute of Technology and Science Pilani**
**First Semester 2022-23**
**Ordinary Differential Equation (MATH F312)**
**Mid-Semester Examination (Closed Book)**

**Time: 90 minutes** **Date: 05-11-2022** **Max Marks: 60**

**Q1.** Let $g(t,u)$ be a continuous function and satisfies Lipschitz condition in a closed and bounded region $R = \{(t,u) | |t - t_0| \leq a \text{ and } |u - u_0| \leq b\}$, then prove that there exist a unique solution $u(t)$ to the initial value problem $u' = g(t,u),\ u(t_0) = u_0$ defined on the interval $|t - t_0| \leq h$, where $h = \min(a, b/M)$ and $|g(t,u)| \leq M$ on $R$. **[18]**

**Q2.** Let $c \geq 0$ be a constant and $u(t) \geq 0, v(t) \geq 0$, be continuous functions on some interval $[t_0, t_0 + a],\ a > 0$ satisfying

$$u(t) \leq c + \int_{t_0}^{t} u(s)v(s)ds, \quad t \in [t_0, t_0 + a].$$

Then prove that the inequality

$$u(t) \leq c \exp\left(\int_{t_0}^{t} v(s)ds\right) \quad t \in [t_0, t_0 + a].$$

holds. **[8]**

**Q3.** Find the fundamental matrix for the system **[12]**

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 & -3 \\ 1 & 1 & 2 \\ 1 & -1 & 4 \end{bmatrix} \mathbf{x}.$$

**Q4.** Prove or disprove that for all the solutions $\mathbf{x}(t)$ of the system

$$\mathbf{x}' = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -10 & -11 & -5 \end{bmatrix} \mathbf{x}.$$

$\lim_{t \to \infty} \|\mathbf{x}(t)\| = 0.$ **[5]**

**Q5.** Let $\Phi$ be a fundamental matrix for the system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$**,** where $\mathbf{A}(t)$ is a continuous matrix and C is a constant nonsingular matrix. Prove that $\Phi$C is also a fundamental matrix of the above system and further prove that any fundamental matrix of this system will be of the form $\Phi$D, where D is a constant nonsingular matrix (The order of $\mathbf{A}, \Phi, \mathrm{C}, \mathrm{D}$ are $n$). **[8]**

**Q6.** Using Picard's iterative method, find first three (in addition to $\mathbf{x_0} = (1, -1)$) non-zero approximations to the IVP **[9]**

$$x_1' = t + x_2$$
$$x_2' = t + x_1$$
$$x_1(0) = 1, \quad x_2(0) = -1.$$

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