Birla Institute of Technology and Science Pilani
First Semester 2022-23
Ordinary Differential Equation (MATH F312)
Comprehensive Examination (Close Book)

Time: 100 minutes **Date: 30-12-2022** **Max Marks: 50**

Q1. (i) Define stability and asymptotic stability of any solution $x(t) = x(t, t_0, x_0)$, of the system $\mathbf{x}' = \mathbf{F}(t, \mathbf{x})$ **[2]**

(ii) Prove that if all the characteristic roots of the $n \times n$ constant matrix A have negative real parts, then every solution of the system $\mathbf{x}' = A\mathbf{x}$, is asymptotically stable. **[5]**

(iii) Prove that the zero solution of $u' = -\alpha u,\ \alpha > 0$ is asymptotically stable. **[3]**

Q2. Let $a(t)$ be a continuously differentiable, positive function on $[0, \infty)$. Then prove that all the solutions of $u'' + a(t)u = 0$, are bounded on $[0, \infty)$ provided $a(t) \to \infty$ monotonically as $t \to \infty$. **[8]**

Q3. (i) Prove that the nontrivial solutions of $u'' + (1 + \phi(t))u = 0$, where $\phi(t) \to 0$ as $t \to \infty$ are oscillatory. **[4]**

(ii) Prove or disprove that the differential equation $u'' + \frac{1}{4t^2}u = 0$, is nonoscillatory. **[3]**

Q4. For the system

$$x_1' = (x_1 - 1)(x_1 + x_2)$$
$$x_2' = x_2 - x_1^2$$

(i) Find all the critical points.

(ii) Use linearization approach to conclude whether the critical points are stable, unstable or asymptotically stable. **[2+9]**

Q5. Using the Krasovskii's method, prove that the zero solution of the system

$$x_1' = -x_1$$
$$x_2' = x_1 - x_2 - x_2^3,$$

is asymptotically stable. **[5]**

Q6. Find all possible limit cycles and discuss its stability behavior for the following system of differential equations **[9]**

$$x_1' = x_2 + x_2(x_1^2 + x_2^2 - 1)(x_1^2 + x_2^2 - 2)$$
$$x_2' = -x_1 + x_2(x_1^2 + x_2^2 - 1)(x_1^2 + x_2^2 - 2).$$

********END********

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Comprehensive Examination (Open Book)

Time: 80 minutes **Date: 30-12-2022** **Max Marks: 40**

Q1. Using the Gronwall-Reid-Bellman inequality, prove the existence of a unique solution on $J = [t_0, t_0 + a]$, to the initial value problem

$$u''' + g(t)u'' + h(t)u' = 0, \qquad u(t_0) = u_0, u'(t_0) = u_1, u''(t_0) = u_2,$$

where, $g(t), h(t)$ are continuous functions on J and a is a fixed positive real number. **[8]**

Q2. Solve the differential equation $y''' - 3y' - 2y = 3t,\ t \geq 0 = t_0$ by using the wronskian approach. **[8]**

Q3. For the system $(t > 0)$

$$x_1' = -x_1 + (2 + e^{-t})x_2 + x_3$$
$$x_2' = \frac{1}{t}x_1 - \left(2 + \frac{3}{t^2}\right)x_2 + \frac{(2t + 3t^2 e^{-t})}{t}x_3$$
$$x_3' = \frac{1 + 2t}{t^2}x_2 + \left(\frac{1}{t^2} - 1\right)x_3,$$

prove or disprove that for all the solutions $\mathbf{x}(t)$, $\lim_{t\to\infty}\|\mathbf{x}(t)\| = 0$. Use $\|A\| = \sum\left|a_{ij}\right|, 1 \leq i, j \leq n$ wherever needed. **[8]**

Q4. Prove (without solving the equation) that there exist a unique solution to the initial value problem,

$$u' = u^2 - \cos^2 t\,, u(0) = 0,$$

and also find the interval of existence of solution given that $R = \{(t, u) | 0 \leq t \leq a \text{ and } |u| \leq b\}$, where $a > 1, b > 0$. **[8]**

Q5. Find a fundamental matrix for the system **[8]**

$$x_1' = x_1 + x_2$$
$$x_2' = 4x_1 - 2x_2,$$

and hence find a fundamental matrix for the system

$$x_1' = -x_1 - 4x_2$$
$$x_2' = -x_1 + 2x_2.$$

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