Birla Institute of Technology and Science Pilani
First Semester 2023-24
Ordinary Differential Equation (MATH F312)
Mid-Semester Examination (Closed Book)

Time: 90 minutes **Date: 09-10-2023** **Max Marks: 65**

Q1. Let $g(t,u)$ and $\frac{\partial g}{\partial u}$ be continuous on $R = \{(t,u) \,|\, |t - t_0| \le a \text{ and } |u - u_0| \le b\}$, then prove that $g(t,u)$ satisfies Lipschitz condition on R. Is the converse true? Justify your answer. **[8]**

Q2. For an initial value problem (IVP), $u' = g(t,u),\ u(t_0) = u_0$, prove that

$$|u_n(t) - u(t)| \le ML^n \frac{(t - t_0)^{n+1}}{(n+1)!}, \quad \text{on } [t_0, t_0 + \alpha], \quad \alpha > 0,$$

where, $u_n(t)$ is the Picard's n^{th} iteration and $u(t)$ is the exact solution of the IVP. Furthermore, $|g(t,u)| \le M$ and L is Lipschitz constant for $g(t,u)$ on $R = \{(t,u) \,|\, |t - t_0| \le a \text{ and } |u - u_0| \le b\}$. **[9]**

Q3. Let $f(t)$ be a continuous function and $v(t)$ be a nonnegative continuous function on some interval $[t_0, t_0 + a],\ a > 0$. If a continuous function $u(t)$ satisfies

$$u(t) \le f(t) + \int_{t_0}^{t} u(s)v(s)ds, \quad t \in [t_0, t_0 + a].$$

Then prove that the inequality

$$u(t) \le f(t) + \int_{t_0}^{t} f(s)v(s) \exp\left(\int_{s}^{t} v(\tau)d\tau\right) ds, \quad t \in [t_0, t_0 + a].$$

holds. **[12]**

Q4. Consider the IVP:

$$u' = t^2u^2 + 2u^4, \quad u(0) = 2 = u_0,$$

on $R = \{(t,u): 0 \le t \le 1, |u - 2| \le 3\}$, then estimate the variation in its solution for $0 \le t \le 1$ when u_0 is perturbed by 0.01. **[7]**

Q5. Let Φ be a fundamental matrix of the system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$, where $\mathbf{A}(t)$ is a continuous square matrix of order n, then prove that

$$W'(t) = \left(Trace\ \mathbf{A}(t)\right)W(t),$$

where, $W(t) = \det \Phi(t)$. **[12]**

Q6. Find the exponential matrix for the system

$$\mathbf{x}' = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}\mathbf{x},$$

and hence find a fundamental matrix for the system $\boldsymbol{y}' = \begin{bmatrix} -1 & 1 \\ -4 & 3 \end{bmatrix}\mathbf{y}$. **[11]**

Q7. Prove that for a system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$, where $\mathbf{A}(t)$ is a continuous square matrix of order n, a fundamental system of solutions exist. **[6]**

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