Birla Institute of Technology and Science Pilani
First Semester 2023-24
Ordinary Differential Equation (MATH F312)
Comprehensive Examination (Close Book)

Time: 100 minutes **Date: 07-12-2023** **Max Marks: 50**

Q1. (i) Prove that all the solutions of $\mathbf{x}' = A(t)\mathbf{x}$ will be stable iff they are bounded. Here, $A(t)$ is an $n \times n$ continuous matrix on $[0, \infty)$. **[8]**

(ii) Prove or disprove that the zero solution of

$$\mathbf{x}'(t) = \begin{bmatrix} -1 & e^{2t} \\ 0 & -3 \end{bmatrix} \mathbf{x}(t).$$

is stable on $[0, \infty)$. Use $\|(x_1, x_2)\| = |x_1| + |x_2|$. **[6]**

Q2. Let $b(t)$ be a continuously differentiable function on $[0, \infty)$. Then prove that all the solutions of $u'' + \big(1 + b(t)\big)u = 0$, are bounded on $[0, \infty)$ provided $b(t) \to 0$ as $t \to \infty$ and $\int_0^\infty |b'(s)| < \infty$. Furthermore, discuss the boundedness of solutions of $u'' + \left(1 + \frac{1}{(t+1)^2} + \frac{1}{(t+1)}\right)u = 0.$ **[12]**

Q3. (i) Let $a(t), b(t)$ be a continuous real valued function on $[t_0, \infty)$ such that $a(t) \leq b(t)$ on $[t_0, \infty)$. Prove that if all the non-trivial solutions of $u'' + a(t)u = 0$, are oscillatory on $[t_0, \infty)$, then all the non-trivial solutions of $v'' + b(t)v = 0$, will be oscillatory on $[t_0, \infty)$. **[8]**

(ii) Prove or disprove that any nontrivial solution of differential equation $u'' - \frac{1}{t^4}u = 0$, vanish more than once in $(1, 4)$. **[3]**

Q4. Prove or disprove that for all the solutions $\mathbf{x}(t)$ of the system

$$\mathbf{x}' = \begin{bmatrix} -5 & 4 & 0 \\ 0 & 0 & 3 \\ -12 & 8 & -6 \end{bmatrix} \mathbf{x},$$

$\lim_{t \to \infty} \|\mathbf{x}(t)\| = 0.$ **[5]**

Q5. If all the characteristic roots of the characteristic polynomial of the differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0,$$

where $a_1, a_2, \ldots, a_n$ are the real constants, have negative real parts, then prove that for any solution $y(t)$ of above equation, there exist positive numbers α, M such that $|y(t)| \leq M e^{-\alpha t}$, $t \geq 0$. **[8]**

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Time: 80 minutes **Date: 07-12-2023** **Max Marks: 40**

Q1. Using the Gronwall-Reid-Bellman inequality, prove the existence of a unique solution on $J = [t_0, t_0 + a]$, to the initial value problem

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + B(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0,$$

where, $A(t), B(t)$ are continuous matrices of order $n \times n$ and $n \times 1$ respectively on J and $\mathbf{x}(\mathrm{t}), \mathbf{x_0}$ are column vectors. **[4]**

Q2. Compute the Picard's first three iterations for the IVP $u' = 1 + tu,\ u(0) = 2$. **[6]**

Q3. Using a suitable Liapunov function of the form $ax_1^2 + bx_2^2$, for appropriate choices of a, b, to prove the following:

(i) zero solution for the system

$$x_1' = 2x_1x_2 + x_1^3$$
$$x_2' = -x_1^2 + x_2^5,$$

is unstable.

(ii) zero solution for the system

$$x_1' = -x_1^3 + x_1^2x_2$$
$$x_2' = -x_1^3 - x_1^2x_2,$$

is stable. **[4+4]**

Q4. Investigate the stability based on first approximation method of zero solution of the system **[7]**

$$x_1' = -2x_1 + 5x_2 + 2x_2^3$$
$$x_2' = -4x_1 - 2x_2 + 3x_1^3,$$

Q5. Using the Krasovskii's method, prove that the zero solution of the system

$$x_1' = -3x_1 - 2x_2 - x_1^5$$
$$x_2' = 2x_1 - 3x_2 - x_2^5,$$

is asymptotically stable. **[7]**

Q6. Find all possible limit cycles and discuss its stability behavior for the following system of differential equations **[8]**

(i)
$$x_1' = -x_2 + x_1(1 - x_1^2 - x_2^2)$$
$$x_2' = x_1 + x_2(1 - x_1^2 - x_2^2).$$

(ii)
$$x_1' = x_2 + x_1(4 - x_1^2 - x_2^2)^2$$
$$x_2' = -x_1 + x_2(4 - x_1^2 - x_2^2)^2.$$

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