A **PART A**

Birla Institute of Technology and Science, Pilani
Second Semester 2022-23, Comprehensive Examination (Closed Book)
MATH F 341: Introduction to Functional Analysis

Max. Marks: 45 Date: 17.05.2023 Time: 60 mins

Name: ID. No.

Write the correct choice (A, B, C or D) in the following table. Each correct answer carries 3 marks and incorrect answer –1 mark.

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
ANS															

1. Let $E = \{e_1,\ e_2,\ e_3\}$ be an orthonormal set in a Hilbert space H. Then which of the following is true?
   (A) $\|e_1 - e_2 - e_3\| = 0.$
   (B) $\|e_1 - e_2 - e_3\| = 3$
   (C) $\|e_1 - e_2 - e_3\| = \sqrt{3}.$
   (D) None of the above

2. Let M be a closed subspace of a normed linear space X, and $T: X \to \frac{X}{M}$ be the natural mapping. Then which of the following is true?
   (A) $\|T\| \leq 1$
   (B) T is discontinuous at zero
   (C) T is continuous but not linear
   (D) None of the above

3. Which of the following is a true statement?
   (A) l^1 and l^∞ both are separable
   (B) l^1 is separable but l^∞ is not separable
   (C) l^1 is not separable but l^∞ is separable
   (D) None of the above

4. Let T be a bounded linear transformation from a normed linear space X onto a normed linear space Y. If there is a positive real number b such that $\|Tx\| \geq b\|x\|$ for all $x \in X$. Then which of the following is true?
   (A) T^{-1} exists but T^{-1} is not bounded
   (B) T^{-1} does not exists
   (C) T^{-1} exists and T^{-1} is bounded
   (D) None of the above

5. X be an infinite dimensional normed linear space, $E = \{x \epsilon X \colon \|x\| \leq 1\}\ and$ $F = \{x \epsilon X \colon \|x\| = 1\}$. Then which of the following is true?
   (A) E and F both are not compact.
   (B) E and F both are compact.
   (C) E is compact but F is not compact
   (D) F is compact but E is not compact

6. Let T be a continuous linear functional defined on a normed linear space X, and $N(T)$ denotes the null space of T. Then which of the following is true?
   (A) $N(T)$ is open
   (B) $N(T)$ is closed
   (C) $N(T)$ is neither open nor closed
   (D) None of the above

7. Consider the following two statements:
S1: Every linear operator on a finite-dimensional normed linear space is continuous.
S2: Any two norms on a finite-dimensional normed linear space are equivalent.
Then which of the following is true?
(A) S1 is true and S2 is false (B) S1 is false and S2 is true
(C) S1 and S2 both are false (D) S1 and S2 both are true

8. Let $T \in B(H)$ be self adjoint. Then which of the following is true?
(A) $\langle Tx, x \rangle$ is real for all $x \in H$ (B) $\langle Tx, x \rangle$ is purely imaginary for all $x \in H$
(C) T is not normal (D) None of the above

9. Let $T \in B(H)$ and K be an eigenvalue of T. Then which of the following is true?
(A) $|K| > \|T\|$ (B) $|K| = 1$
(C) $|K| \leq \|T\|$ (D) None of the above

10. Let Y be a subspace of an inner product space X. Let $x\epsilon X$ satisfies $\|x - y\| \geq \|x\|$ for all $y\epsilon Y$. Then which of the following is true?
(A) $\langle x, y \rangle > 1$ for all $y\epsilon Y$ (B) $0 < \langle x, y \rangle < 1$ for all $y\epsilon Y$
(C) $\langle x, y \rangle = 1$ for all $y\epsilon Y$ (D) None of the above

11. Let $T \in B(H)$. Then which of the following is true?
(A) T^*T is self adjoint and TT^* is normal (B) T^*T is not self adjoint and TT^* is normal
(C) T^*T is self adjoint and TT^* is not normal (D) None of these

12. Let H be a complex Hilbert space and $A \in B(H)$ be a normal operator. Let $r_A =$ spectral radius of A, $R_A =$ numerical radius of A. Then which of the following is true?
(A) $r_A < R_A \leq \|A\|$ (B) $r_A = R_A < \|A\|$
(C) $r_A = R_A = \|A\|$ (D) None of the above

13. Let $T \in B(l^2)$ be defined by $T(x_1, x_2, x_3, x_4, \ldots \ldots) = (0, x_1, x_2, x_3, x_4, \ldots \ldots)$, and $e(T) =$ set of all eigenvalues of A. Then which of the following is true?
(A) $e(T) = \{1\}$ (B) $e(T) = \{-1\}$
(C) $e(T) = \{1, -1\}$ (D) None of the above

14. Let S be a subset of a Hilbert space H. Which of the following statements is FALSE?
(A) $S^{\perp} = (\bar{S})^{\perp}$ (B) $S^{\perp}$ is a closed subspace of H
(C) $S^{\perp\perp}$ is the smallest subspace of Hcontaining S (D) None of these

15. Consider the following statements:
S1: Let T be linear transformation from a normed linear space X to a normed linear space Y, and T be
continuous at $x = 0$. Then T need not be continuous at every point of X.
S2: Let Y be subspace of a normed linear space X, and $dimY < \infty$. Then Y must be closed in X.
Then which of the following is true?
(A) S1 is true and S2 is false (B) S1 is false and S2 is true
(C) S1 and S2 both are false (D) S1 and S2 both are true

****

SET B **PART A**

Birla Institute of Technology and Science, Pilani
Second Semester 2022-23, Comprehensive Examination (Closed Book)
MATH F 341: Introduction to Functional Analysis

Max. Marks: 45 Date: 17.05.2023 Time: 60 mins

Name: ID. No.

Write the correct choice (A, B, C or D) in the following table. Each correct answer carries 3 marks and incorrect answer –1 mark.

Q.No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
ANS															

1. Let $T \in B(l^2)$ be defined by $T(x_1, x_2, x_3, x_4, \dots \dots) = (0, x_1, x_2, x_3, x_4, \dots \dots)$, and $e(T)$ = set of all eigenvalues of A. Then which of the following is true?
   (A) $e(T) = \{1\}$ (B) $e(T) = \{-1\}$
   (C) $e(T) = \{1, -1\}$ (D) None of the above

2. Let S be a subset of a Hilbert space H. Which of the following statements is FALSE?
   (A) $S^{\perp} = (\bar{S})^{\perp}$ (B) $S^{\perp}$ is a closed subspace of H
   (C) $S^{\perp\perp}$ is the smallest subspace of Hcontaining S (D) None of these

3. Consider the following statements:
   S1: Let T be linear transformation from a normed linear space X to a normed linear space Y, and T be
   continuous at $x = 0$. Then T need not be continuous at every point of X.
   S2: Let Y be subspace of a normed linear space X, and $dimY < \infty$. Then Y must be closed in X.
   Then which of the following is true?
   (A) S1 is true and S2 is false (B) S1 is false and S2 is true
   (C) S1 and S2 both are false (D) S1 and S2 both are true

4. Let $E = \{e_1, e_2, e_3\}$ be an orthonormal set in a Hilbert space H . Then which of the following is true?
   (A) $\|e_1 - e_2 - e_3\| = 0.$ (B) $\|e_1 - e_2 - e_3\| = 3$
   (C) $\|e_1 - e_2 - e_3\| = \sqrt{3}.$ (D) None of the above

5. Let M be a closed subspace of a normed linear space X, and $T: X \to \frac{X}{M}$ be the natural mapping. Then which of the following is true?
   (A) $\|T\| \leq 1$ (B) T is discontinuous at zero
   (C) T is continuous but not linear (D) None of the above

6. Which of the following is a true statement?
   (A) l^1 and l^∞ both are separable
   (B) l^1 is separable but l^∞ is not separable
   (C) l^1 is not separable but l^∞ is separable
   (D) None of the above

7. Let T be a bounded linear transformation from a normed linear space X onto a normed linear space Y. If there is a positive real number b such that $\|Tx\| \geq b\|x\|$ for all $x \in X$. Then which of the following is true?
   (A) T^{-1} exists but T^{-1} is not bounded
   (B) T^{-1} does not exists
   (C) T^{-1} exists and T^{-1} is bounded
   (D) None of the above

8. X be an infinite dimensional normed linear space, $E = \{x \epsilon X : \|x\| \leq 1\}\ and$ $F = \{x \epsilon X : \|x\| = 1\}$. Then which of the following is true?
   (A) E and F both are not compact.
   (B) E and F both are compact.
   (C) E is compact but F is not compact
   (D) F is compact but E is not compact

9. Let T be a continuous linear functional defined on a normed linear space X, and $N(T)$ denotes the null space of T. Then which of the following is true?
   (A) $N(T)$ is open
   (B) $N(T)$ is closed
   (C) $N(T)$ is neither open nor closed
   (D) None of the above

10. Consider the following two statements:
    S1: Every linear operator on a finite-dimensional normed linear space is continuous.
    S2: Any two norms on a finite-dimensional normed linear space are equivalent.
    Then which of the following is true?
    (A) S1 is true and S2 is false
    (B) S1 is false and S2 is true
    (C) S1 and S2 both are false
    (D) S1 and S2 both are true

11. Let $T \in B(H)$ be self adjoint. Then which of the following is true?
    (A) $\langle Tx, x\rangle$ is real for all $x \in H$
    (B) $\langle Tx, x\rangle$ is purely imaginary for all $x \in H$
    (C) T is not normal
    (D) None of the above

12. Let $T \in B(H)$ and K be an eigenvalue of T. Then which of the following is true?
    (A) $|K| > \|T\|$
    (B) $|K| = 1$
    (C) $|K| \leq \|T\|$
    (D) None of the above

13. Let Y be a subspace of an inner product space X. Let $x \epsilon X$ satisfies $\|x - y\| \geq \|x\|$ for all $y \epsilon Y$. Then which of the following is true?
    (A) $\langle x, y\rangle > 1$ for all $y \epsilon Y$
    (B) $0 < \langle x, y\rangle < 1$ for all $y \epsilon Y$
    (C) $\langle x, y\rangle = 1$ for all $y \epsilon Y$
    (D) None of the above

14. Let $T \in B(H)$. Then which of the following is true?
    (A) T^*T is self adjoint and TT^* is normal
    (B) T^*T is not self adjoint and TT^* is normal
    (C) T^*T is self adjoint and TT^* is not normal
    (D) None of these

15. Let Hbe complex Hilbert space and $A \in B(H)$ be a normal operator. Let $r_A =$ spectral radius of A, $R_A =$ numerical radius of A. Then which of the following is true?
    (A) $r_A < R_A \leq \|A\|$
    (B) $r_A = R_A < \|A\|$
    (C) $r_A = R_A = \|A\|$
    (D) None of the above

*****

PART B

Birla Institute of Technology and Science, Pilani
Second Semester 2022-23, Comprehensive Examination (**Open Book**)
MATH F 341: Introduction to Functional Analysis

Max. Marks: 45 Date: 17.05.2023 Time: 120 mins

Q1. Let $T \in B(H)$ be self adjoint operator and $T \geq 0$. Then show that range space of $I + T$ is closed. [5]

Q2. Let $T \in B(H)$ be a normal operator and $Ker(T) = \{0\}$. Then answer the following question?

(i) Is T^* one-one? Justify your answer.

(ii) Prove that range space of T is dense in H. [3+5=8]

Q3. Let H be complex Hilbert space and $U \in B(H)$ be unitary. Let $f: B(H) \to B(H)$ be a linear operator defined by $f(T) = U^*TU$. Then show that f is an isometry. [$S \in B(H)$ is said to be an isometry if $\|Sx\| = \|x\|$ for all $x \in H$]. [4]

Q4. Let $H = l^2$ be a Hilbert space and $T,\ T^* \in B(H)$ be defined by

$$T(x_1, x_2, x_3, \dots \dots) = (0, x_1, x_2, x_3, \dots \dots),$$
$$T^*(x_1, x_2, x_3, \dots \dots) = (\ x_2, x_3, x_4, \dots \dots).$$

Then answer the following questions:

(i) Find $e(T)$, the set of all eigenvalues of T.

(ii) Show that $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all $x, y \in H$.

(iii) Show that 0 is an eigenvalue of TT^* and find an eigenvector of TT^* corresponding to the eigenvalue 0. [3×3=9]

Q5. Let $H = \mathbb{R}^2$ be a real Hilbert Space and $A \in B(H)$. For $x = (x_1,\ x_2)$, define $Ax = (x_2, -x_1)$. Then answer the following questions:

(i) Is A unitary? Is A normal? Justify your answer in each case.

(ii) Compute R_{A^*} *and* $R_{(A^*)^2}$, where R_{A^*} = numerical radius of A^*. [2×4=8]

Q6. Let H be a real Hilbert Space and $y, z \in H$. Let T be a bounded linear operator defined by

$$T(x) = \langle x, y \rangle z, \quad x \in H.$$

Then show that (i) T is compact, and (ii) $T^*(w) = \langle w, z \rangle y$. [3+4=7]

Q7. Let $E = \{e_1, e_2, \dots \dots, e_n\}$ be an orthonormal set in a Hilbert space H.
Then prove or disprove: E is linearly independent. [4]

END