Birla Institute of Technology and Science, Pilani

2nd Semester 2016-17

Differential Geometry (MATH F342)

Comprehensive Examination

Part I (Closed Book)

Max. Time :100 mins **Max. Marks: 50**

Instructions : 1) Use arrow notation on top of all vectors. 2) All notations used in question paper are standard. 3) Explain all the steps.

1. By applying isoperimetric inequality to a suitable ellipse, show that $\int_0^{2\pi} \sqrt{\cos^2 t + 4\sin^2 t}\, dt > (2\sqrt{2})\pi$.

   [6]

2. Let $U_1 = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 < 1\}$, $U_2 = \{(\theta,\varphi) \in \mathbb{R}^2 : \theta \in (0,2\pi), \varphi \in (0,\pi)\}$ and let $\vec{\sigma}_1 : U_1 \to \mathbb{R}^3$ and $\vec{\sigma}_2 : U_2 \to \mathbb{R}^3$ be the surface patches of the unit sphere $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ defined by $\vec{\sigma}_1(u,v) = \left(u, v, \sqrt{1-u^2-v^2}\right)$ and $\vec{\sigma}_2(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$. Find the transition map from $\vec{\sigma}_1$ to $\vec{\sigma}_2$. Justify your answer. [8]

3. Let $\vec{\sigma}(u,v) = \left(\frac{u}{\sqrt{u^2+v^2+4}}, \frac{v}{\sqrt{u^2+v^2+4}}, \frac{2}{\sqrt{u^2+v^2+4}}\right)$; $(u,v) \in \mathbb{R}^2$ be a regular surface patch of a smooth surface S.
   (A) Find a level surface $f(x,y,z) = 0$ containing $\mathrm{Im}(\vec{\sigma})$.
   (B) Find $(u_0, v_0) \in \mathbb{R}^2$ such that $\vec{\sigma}(u_0,v_0) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.
   (C) Find a basis for the tangent space to S at $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.
   (D) Find the standard surface normal to $\vec{\sigma}$ at $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. [2+3+4+2]

4. Let $\vec{\sigma} : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times (0, \pi) \to \mathbb{R}^3$ be a regular surface patch given by $\vec{\sigma}(\theta,\varphi) = (\cos\varphi, \sin\varphi, \sin\theta)$. Let $f : Im(\vec{\sigma}) \to \mathbb{R}^3$ be defined by $f(x,y,z) = \left(x\sqrt{1-z^2}, y\sqrt{1-z^2}, z\right)$. Answer with justification the following :
   (A) Find the 1st Fundamental forms of $\vec{\sigma}$ and $f \circ \vec{\sigma}$
   (B) Decide if f is an isometry of $\mathrm{Im}(\vec{\sigma})$ with $\mathrm{Im}(f \circ \vec{\sigma})$.
   (C) Is f conformal?
   (D) Is f equiareal? [5+2+2+2]

5. Find the principal curvatures at an arbitrary point of the surface patch whose 1st Fundamental form is $(1+v^2)du^2 + dv^2$ and 2nd Fundamental Form is $\frac{2dudv}{\sqrt{1+v^2}}$, $(u,v) \in (0,1) \times (0,2\pi)$. Hence decide if $\vec{\sigma}\left(\frac{1}{2}, \pi\right)$ is elliptic or hyperbolic or parabolic or planar.

   [6]

6. Let $\vec{\sigma}(u,v) = \left((2+\cos v)\cos u,\ (2+\cos v, v)\sin u, \sin v\right), 0 < u < 2\pi, 0 < v < 2\pi$ be a regular surface patch. Find the 1st Fundamental Form of $\vec{\sigma}$ and hence the surface area of $\mathrm{Im}(\vec{\sigma})$.

   [8]

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Birla Institute of Technology and Science, Pilani

2nd Semester 2016-17

Differential Geometry (MATH F342)

Comprehensive Examination

Part II (Open book)

Max. Time: 80 mins **Max. marks : 40**

Instructions : 1) Use arrow notation on top of all vectors. 2) All notations used in question paper are standard. 3) Explain all the steps.

1. Let $\vec{\gamma}(t) = (a \sin \varpi t, b \sin t); t \in \mathbb{R}$ be a parametrized curve where a, b, ϖ are real constants with $ab\varpi \neq 0$. Show that $\vec{\gamma}$ is regular if and only if ϖ is not a ratio of odd integers. [8]
2. Let $\vec{\gamma}(s)$ be a generalized helix with arc length parameter s and let $\vec{\gamma}^*(s) = \vec{\gamma}'(s)$. Show that $\vec{\gamma}^*(s)$ is a circle. [8]
3. Let $\vec{\gamma}(t) = (x(t), y(t), z(t)); t \in (\alpha, \beta)$ be a smooth unit speed curve such that $\|\vec{\gamma}(t)\| = 1$ for all $t \in (\alpha, \beta)$. Show that $\vec{\gamma}''(t)$ lies in the plane spanned by $\vec{\gamma}(t)$ and $\vec{\gamma}(t) \times \vec{\gamma}'(t)$. Hence show that curvature of $\vec{\gamma}(t)$ is given by $\kappa(t) = \sqrt{1 + J(t)^2}$ where $J(t) = \begin{vmatrix} x(t) & y(t) & z(t) \\ x'(t) & y'(t) & z'(t) \\ x''(t) & y''(t) & z''(t) \end{vmatrix}$. [8]
4. Find a smooth function $f(x)$ for $0 < x < \pi/2$ such that $f''(x) > 0 \ \forall \ x \in (0, \pi/2)$ and for any $0 < x < \pi/2$, the curvature of the curve $y = f(x)$ at $(x, f(x))$ is $\dfrac{1}{\sqrt{1 + f'(x)^2}}$. [8]
5. Let $\vec{\gamma}(t) = (\cos t + 2\sin t, \cos t - 2\sin t), t \in \mathbb{R}$. Find the maximum value of the curvature $\kappa(t)$ of $\vec{\gamma}(t)$. Hence show that the curve $\overrightarrow{\gamma^{\lambda}}(t) = \vec{\gamma}(t) + \lambda \overrightarrow{n_s}(t)$ is regular if $|\lambda| < \dfrac{1}{\sqrt{2}}$, where $\overrightarrow{n_s}(t)$ is the signed unit normal of $\vec{\gamma}(t)$ and λ is constant. [8]

------------------------------------All the best--