Birla Institute of Technology and Science, Pilani

2nd Semester 2017-18

Differential Geometry (MATH F342)

Comprehensive Exam

Part A (Closed book)

Max. Time: 100 mins **Max. marks : 50**

Instructions : 1) Use arrow notation on top of all vectors. 2) All notations used in question paper are standard. 3) Explain all the steps.

1. Let $\vec{\gamma}(t)$ be a simple closed smooth unit speed plane curve with period l and $\kappa_s(t)$ denote its signed curvature. Show that $\int_0^l \frac{d\kappa_s}{dt}\vec{\gamma}(t)dt = \vec{0}$. [8]

2. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z-1)^2 = 1\}$ and $N = (0, 0, 2) \in S$. Find a surface patch $\vec{\sigma}: \mathbb{R}^2 \to S$ such that $\vec{\sigma}(u, v) = (S \cap L) - \{N\}$ where L is the line segment joining $(u, v, 0)$ to N. Also find an open set W of $\mathbb{R}^3$ such that $\vec{\sigma}(\mathbb{R}^2) = S \cap W$. [8]

3. Let $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Let $\vec{\sigma_i}: U_i \to S, i = 1, 2$ be surface patches of S defined by

$$\vec{\sigma_1}(u_1, v_1) = \left(u_1, v_1, \sqrt{1 - u_1^2 - v_1^2}\right), (u_1, v_1) \in U_1 = \{(u_1, v_1) \in \mathbb{R}^2 : u_1^2 + v_1^2 < 1\},$$

$$\vec{\sigma_2}(u_2, v_2) = \left(u_2, \sqrt{1 - u_2^2 - v_2^2}, v_2\right), (u_2, v_2) \in U_2 = \{(u_2, v_2) \in \mathbb{R}^2 : u_2^2 + v_2^2 < 1\}.$$

Find the transition map from $\vec{\sigma_1}$ to $\vec{\sigma_2}$ along with its domain. [8]

4. For the regular surface patch $\vec{\sigma}(u, v) = ((2 + \cos u)\cos v, (2 + \cos u)\sin v, \sin u); (u, v) \in (0, 2\pi) \times (0, 2\pi)$, find the 1st Fundamental Form and hence find the area of the image of $\vec{\sigma}$. [8]

5. Let $S_1 = \{(x, y, 0) : x, y \in \mathbb{R}\}$ and $S_2 = \left\{\left(x, y, \sqrt{x^2 + y^2}\right) : (x, y) \in \mathbb{R}^2 - \{(0, 0)\}\right\}$. For the diffeomorphism $f: S_1 \to S_2$ given by $f(x, y, 0) = \left(x, y, \sqrt{x^2 + y^2}\right)$, verify if f is (a) Isometry, (b) conformal map, (c) equiareal map. [10]

6. Suppose a regular surface patch $\vec{\sigma}$ has 1st Fundamental form $(2 + v^2)^2 du^2 + 2uvdudv + (2 + u^2)^2 dv^2$ and the 2nd Fundamental Form $\frac{dudv}{2+u^2+v^2}$.
   (A) Find the normal curvature of $\vec{\gamma}(t) = \vec{\sigma}(t, e^t)$ at $\vec{\gamma}(0)$.
   (B) Find the principal curvatures of $\vec{\sigma}$ at $\vec{\sigma}(0, 1)$.
   (C) Decide whether $\vec{\sigma}(0, 1)$ is elliptic, hyperbolic, parabolic or planar. [2+4+2]

Birla Institute of Technology and Science, Pilani

2nd Semester 2017-18

Differential Geometry (MATH F342)

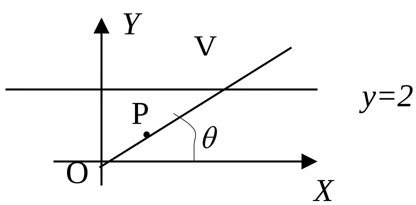
Comprehensive Exam

Part B (Open book)

Max. Time: 80 mins **Max. marks : 40**

Instructions : 1) Use arrow notation on top of all vectors. 2) All notations used in question paper are standard. 3) Explain all the steps.

1. Let V be a point on the line $y = 2$ in $xy-$plane and let P be the point on the line segment from V to the origin O such that $\|\overrightarrow{VP}\| = 2$. Find the parametric equation $\vec{\gamma}$ of the curve traced by P as V varies over the line $y = 2$ with parameter $\theta =$ the angle made by the line OV with the positive x-axis. Refer to the figure below. What are the values of $\theta \in (0, \pi)$ for which $\vec{\gamma}$ is regular? [6]



2. Find a plane unit speed smooth curve whose signed curvature is given by $\kappa_s(s) = -\frac{1}{s}, s > 0.$ [8]

3. Find the involute $\vec{\iota}(t)$ of $\vec{\gamma}(t) = \left(\frac{t^2}{2}, \frac{t^3}{3}\right), 0 < t < 1.$ [6]

4. Let $\vec{\gamma}(s)$ be smooth unit speed curve in $\mathbb{R}^3$ with non-vanishing constant curvature κ and non-vanishing torsion. Let $\overrightarrow{\gamma_1}(s) = \vec{\gamma}(s) + \frac{\vec{n}(s)}{\kappa}$. Find the curvature and torsion of $\overrightarrow{\gamma_1}(s)$. [10]

5. Suppose $\vec{\gamma}(s)$ is a smooth unit speed space curve with non-vanishing curvature $\kappa(s)$ and torsion $\tau(s)$. Let $\frac{d^n\vec{\gamma}}{ds^n} = a_n(s)\vec{t}(s) + b_n(s)\vec{n}(s) + c_n(s)\vec{b}(s)$ for any positive integer n where $\vec{t}, \vec{n}, \vec{b}$ are respectively unit tangent, principal normal and binormal vectors of $\vec{\gamma}$. Show that for any positive integer n, we have $a_{n+1}(s) = \frac{da_n}{ds} - \kappa(s)b_n(s)$, $b_{n+1}(s) = \frac{db_n}{ds} + \kappa(s)a_n(s) - \tau(s)c_n(s)$ and $c_{n+1}(s) = \frac{dc_n}{ds} + \tau(s)b_n(s)$. Hence find $\frac{d^4\vec{\gamma}}{ds^4}$ without evaluating $\frac{d^n\vec{\gamma}}{ds^n}$ for n = 2, 3. [10]