Birla Institute of Technology & Science, Pilani

MATH F353 (Statistical Inference and Applications)
Second Semester 2021-2022
Comprehensive Examination (Open Book)

Time: 180 Minutes **Date: May 09, 2022 (Monday)** **Max. Marks: 80**

Note: Only class-notes are allowed. Notations and symbols have their usual meaning. Define random variables as and when required. Assumptions made if any, should be stated clearly at the beginning. Start new question on fresh page. Moreover, answer each subpart of a question in continuation.

1. Let $X_1, X_2, ..., X_n$ be a random sample from a population with pdf

$$f(x, \theta) = kxe^{-\theta x}; 0 < x < \infty,\ \theta > 0 \text{ and } k \text{ is a real constant.}$$

   (a) Find k such that $f(x, \theta)$ is a valid pdf.

   (b) Find the MLE of θ. Is it unbiased?

   (c) Prove or disprove that $Y = \sum_1^n X_i$ is a complete sufficient statistic for θ.

   (d) Find the MVUE of θ. **[20]**

2. Let $X_1, X_2, ..., X_n$ be a random sample from a population with pdf

$$f(x, \theta) = (1 - \theta)\theta^x, x = 0, 1, 2, ...; 0 \le \theta \le 1.$$

   Does this family of distributions have an MLR in $\left(\theta, -\sum_1^n X_i\right)$? **[5]**

3. Let X be one observation from

$$f(x, \theta) = \frac{1}{\pi}\frac{\theta}{(\theta^2 + x^2)},\ \theta > 0.$$

   Does this family of distributions have an MLR in (θ, X)? **[5]**

4. Let $X_1, X_2, ..., X_n$ be a random sample from a $N(\mu, \sigma^2)$ family, where σ^2 is known. Consider testing $H_0 : \mu \le 0$ versus $H_1 : \mu > 0$. Derive an LRT of size α. Also, obtain the form of the power function. **[10]**

5. The random variable X has the pdf $f(x) = e^{-x}, x > 0$. One observation is obtained on the random variable $Y = X^\theta$, and a test of $H_0 : \theta = 1$ versus $H_1 : \theta = 2$ needs to be constructed. Find the pdf of Y. Hence, derive the UMP level $\alpha = 0.10$ test and compute the Type II Error probability. **[10]**

6. Let $X_1, X_2, ..., X_n$ be a random sample from exponential(λ). Find an unbiased estimator of λ based only on $Y = \min(X_1, X_2, ..., X_n)$. Can you suggest a better estimator than the one you just obtained? **[10]**

7. Let $X_1, X_2, ..., X_n$ be iid $N(\theta, 1)$. Prove or disprove that $T = \bar{X}^2 - (1/n)$ is the best unbiased estimator of θ^2. **[10]**

8. Let $X_1, X_2, ..., X_n$ be a random sample from a $N(\mu, \sigma^2)$ family, where σ^2 is known. Show that $\bar{X}$ and S^2 are independent. **[10]**

END