BIRLA INSTITUTE OF TECHNOLOGY & SCIENCE, PILANI

Comprehensive Exam: Second Semester, 2022-23

Stochastic Processes and their Applications (MATH F424)

May 6, 2023

Max. Time :180 mins **Max. Marks: 45**

Name :.. **ID No. :**..

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1. The number of cars visiting a national park forms a Poisson process with rate 15 per hour. Each car has k occupants with probability p_k as given below:

   $p_1 = 0.2, p_2 = 0.3, p_3 = 0.3, p_4 = 0.1, p_5 = 0.05, p_6 = 0.05.$

   Compute the mean and variance of the number of visitors to the park during a 10-hour window. [5]

2. Customers arrive at a service station according to a poisson process with parameter λ. The service station has two servers.

   (a) An arriving customer is routed to server 1 or server 2 with equal probability. What are the mean and variance of the time between two consecutive arrivals to the first server? [7]

   (b) Next suppose we route the arriving customers to the two servers alternately staring with server 1. Now what are the mean and variance of the time between two consecutive arrivals to the first server? Is the arrival process of the customers to server 1 a Poisson process? [5+3]

3. Consider the following geometric Brownian motion

   $$X(t) = X(0)\exp\left[\sigma W(t) + \left(\alpha - \frac{\sigma^2}{2}\right)t\right]; \quad t > 0$$

   where $W(t)$ is the standard Brownian motion, $\sigma > 0$ be the volatility and α be a constant (risk-free return rate).

   (a) Prove or disapprove that $X(t)$ is a martingale. [10]

   (b) For which value(s) of α, $X(t)$ is a super-martingale/sub-martingale? Justify your answer with common sense reasoning. [3+1]

   (c) Using Ito-Doeblin formula write

   $$dX(t) = \Delta(X(t))\,dW(t) + \Theta(X(t))\,dt$$

   find $\Delta(X(t))$ and $\Theta(X(t))$. [6]

4. Suppose a stock is currently trading at $\$25.00$ and its volatility parameter is $\sigma = 0.5$ per year. Compute the Black–Scholes price of a European call option at strike price $\$15$ and maturity date 6 months. Assume the risk-free return rate is $r = 0.05$ per year. [5]